Probability and Stochastic Processes (2023-24)
Problem Sheet 1

1. Consider the following game, played by throwing three standard six-sided dice in three rounds. If the player ends with all three dice showing the same number, the player wins. The player starts by rolling all three dice. After this first roll, the player can select any one, two, or all of the three dice and re-roll them. After this second roll, the player can again select any of the three dice and re-roll them one final time. For questions (a)–(d), assume that the player uses the following optimal strategy: if all three dice match, the player stops and wins; if two dice match, the player re-rolls the die that does not match; and if no dice match, the player re-rolls them all.

   (a) Find the probability that all three dice show the same number on the first roll.

   (b) Find the probability that exactly two of the three dice show the same number on the first roll.

   (c) Find the probability that the player wins, conditioned on exactly two of the three dice showing the same number on the first roll.

   (d) By considering all possible sequences of rolls, find the probability that the player wins the game.

2. Given a deck of cards, a player draws three cards consecutively. The first card is an Ace of Diamonds, the second card is a Jack of Spades. What is the probability that the next card drawn is a Ace of Spades? Calculate the answer both with and without using conditional probability. Verify if they are same.

3. Show that if $P(A|B) = 1$, then $P(B^c|A^c) = 0$. Also give an example where $P(A|B)$ is close to 1, however $P(B^c|A^c)$ is close to 0.

4. Let's say smokers are 20 times more likely to have lung cancer compared to non-smokers. Also, 25% of the population smokes. Given a person has lung cancer, what is the probability that the person is a smoker

5. We know that mutual independence is a stronger condition compared to pairwise independence. Given an experiment of two consecutive coin tosses, give an example of three random events A, B, C such that they are pairwise independent but not mutually independent.

   [Bonus question: How many such triplet of events are there?]

6. In a office there are three employees shortlisted to be fired, A, B and C. However, the management decides at the last moment that one of them can be retained and chooses the employee to be retained uniformly at random. The decision is communicated to the HR, however they are not allowed to disclose that to the employees. A asks the HR to at least name one amongst B and C who will be fired. The HR is trying to hide as much information as possible, hence

if A is to be retained, the HR will name B and C with equal probability($\frac{1}{2}$). The HR says "B will be fired". Does this help A at all?

[Bonus Question: Can you think of a question for which the same answer by the jailer ["B will be fired"] helps A?]

7. (a) Let A and B be events with $0 < P(A \cap B) < P(A) < P(B) < P(A \cup B) < 1$.You are hoping that both A and B occurred. Which of the following pieces of information would you be happiest to observe: that A occurred, that B occurred, or that $A \cup B$ occurred?
   (b) Given an random experiment where two standard dices are rolled consecutively, construct two events such that the given condition is satisfied. Verify your claim.

8. Suppose two suspects X and Y are equally likely to have committed a crime. However, new evidence found a blood type matching X which is present only in 5% of the population.
   (a) Given this information, what is the probability that X has committed the crime?
   (b) What is the probability that Y's blood type matches the one at crime scene?

9. Suppose there are r red balls and b blue balls in a box. Each time a ball is drawn, k balls of the same colour are added to the box. Let, R_n and B_n be the event that the n-th ball drawn is Red and Blue, respectively. Find $P(R_n)$ and $P(B_n)$.

10. Consider the random experiment of tossing an unbiased coin twice. In this setup, construct three events X, Y, Z such that:
   (a) If X and Y are independent and Y and Z are independent, but X and Z are dependent.
   (b) X and Y are independent, however they are conditionally dependent given Z.
   (c) X and Y are dependent, however they are conditionally independent given Z.

11. Suppose you are playing an online game where the opponent is equally likely to be an amateur or a pro, giving you a chance of success of 80% and 20%, respectively.
   (a) Given you have won the first match, what is the probability that you will win the second match?
   (b) Given you have won at least one match, what is the probability that you were playing a pro?

12. Let $X_1, X_2, ..., X_n$ be random events occurring with probability $p_1, p_2, ..., p_n$, respectively. Show that $P(\cup_{i=1}^{n} X_i) \leq \Sigma_{i=1}^{n} P(X_i)$.

13. Consider a two player (A, B) game where each player tosses a coin and A gets 1 point if it comes up head, and B gets 1 point otherwise. The first one to get to 10 points would win the game. However, the game is interrupted when the score of A and B is 8 and 6, respectively. Calculate the probability of A winning the game if:
   (a) The coin is unbiased.
   (b) The coin comes up head with probability p.

14. Suppose you perform a Bernoulli Trial with an unknown probability of success p repeatedly, resulting in n consecutive successes. What would be your estimate of p?

   [*Hint: Initially, nothing is known about* p*. Assume that* $p \sim U(0, 1)$*.*]

15. Let $\mathcal{A}$ be an algorithm that has a one-sided error and whose failure probability is at most $(1 - 1/\binom{n}{2})$. Show how with independent runs of $\mathcal{A}$, the success probability can be amplified.

16. Suppose you are given a biased coin with unknown probabilities. The goal is to simulate an unbiased coin,i.e. success and fail.

    (a) Think of a basic strategy for the simulation. *[Hint: Try flipping the coin twice.]*

    (b) Given that the biased coin lands on head with probability p, calculate how many number of flips are required in expectation to obtain our outcome?

    (c) Given that the biased coin lands on head with probability $\frac{2}{3}$, can you construct a better(faster) strategy to simulate an unbiased coin?