Probability and Stochastic Processes (2023-24)
Problem Sheet 1

1. Consider the following game, played by throwing three standard six-sided dice in three rounds. If the player ends with all three dice showing the same number, the player wins. The player starts by rolling all three dice. After this first roll, the player can select any one, two, or all of the three dice and re-roll them. After this second roll, the player can again select any of the three dice and re-roll them one final time. For questions (a)–(d), assume that the player uses the following optimal strategy: if all three dice match, the player stops and wins; if two dice match, the player re-rolls the die that does not match; and if no dice match, the player re-rolls them all.

   (a) Find the probability that all three dice show the same number on the first roll.

   (b) Find the probability that exactly two of the three dice show the same number on the first roll.

   (c) Find the probability that the player wins, conditioned on exactly two of the three dice showing the same number on the first roll.

   (d) By considering all possible sequences of rolls, find the probability that the player wins the game.

   Solution: (a) 6 possibilities $\times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$
   (b) $({}^3C_2$ combinations$) \times \big($6 possibilities $\times \frac{1}{6} \times \frac{1}{6}\big) \times \big($5 possibilities for the last $\frac{1}{6}\big) = \frac{15}{36}$
   (c) probability of winning in the second roll is $= \frac{1}{6}$ and prob of winning in third roll $= \frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$. Therefore of total probability $\frac{11}{36}$
   (d) Probability that all 3 are same in the first roll is $\frac{1}{36}$, probability that all 3 are distinct is $\frac{6}{6} \times \frac{5}{6} \times \frac{4}{6} = \frac{20}{36}$.
   We condition on the outcome of first roll. Let S_i denote the event that i dices have same outcome in the first roll.

$$\begin{aligned}\Pr(Win) &= \Sigma_{i=1}^{3} \Pr(Win|S_i) \times \Pr(S_i) \\ &= 1 \times \frac{1}{36} + \frac{11}{36} \times \frac{15}{36} + (1 \times \frac{1}{36} + \frac{11}{36} \times \frac{15}{36}) \times \frac{20}{36}\end{aligned}$$

   We use the fact that three distinct outcomes in the first roll means we re-roll all three dices and thus essentially restart the game.

2. Given a deck of cards, a player draws three cards consecutively. The first card is an Ace of Diamonds, the second card is a Jack of Spades. What is the probability that the next card drawn is a Ace of Spades? Calculate the answer both with and without using conditional probability. Verify if they are same.

3. Show that if $P(A|B) = 1$, then $P(B^c|A^c) = 1$. Also give an example where $P(A|B)$ is close to 1, however $P(B^c|A^c)$ is close to 0.

   Solution: $P(A|B) = 1$ implies $P(A \cap B) = P(B)$ [*Bayes Rule*] or, $B \subseteq A$. Therefore, $P(A^c \cup B^c) = 1$.

4. Let's say smokers are 20 times more likely to have lung cancer compared to non-smokers. Also, 25% of the population smokes. Given a person has lung cancer, what is the probability that the person is a smoker.

   Solution: Use $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

   Let us denote S be the event that a person smokes and L be the event of having lung cancer. We have,

$$
\begin{aligned}
P(S|L) &= \frac{P(L|S)P(S)}{P(L)} \\
&= \frac{P(L|S)P(S)}{P(L|S) \times P(S) + P(L|S^c) \times P(S^c)} && \text{[Using Law of Total Probability]} \\
&= \frac{20p \times 0.25}{20p \times 0.25 + p \times 0.75} && \text{[Assuming } P(L|S^c) = p\text{]} \\
&= \frac{5}{5 + \frac{3}{4}} \\
&= \frac{20}{23}
\end{aligned}
$$

5. We know that mutual independence is a stronger condition compared to pairwise independence. Given an experiment of two consecutive coin tosses, give an example of three random events A, B, C such that they are pairwise independent but not mutually independent.

   [Bonus question: How many such triplet of events are there?]

   Solution: Consider a random experiment by tossing two coins. Consider the three following events

   - E_1 : Coin-1 is tossed Head
   - E_2 : Coin-2 is tossed Tail
   - E_3 : Both coins are tossed same.

   These are pairwise independent (check!) but $P(E_1 \cap E_2 \cap E_3) = 0$.

6. In a office there are three employees shortlisted to be fired, A, B and C. However, the management decides at the last moment that one of them can be retained and chooses the employee to be retained uniformly at random. The decision is communicated to the HR, however they are not allowed to disclose that to the employees. A asks the HR to at least name one amongst B and C who will be fired. The HR is trying to hide as much information as possible, hence if A is to be retained, the HR will name B and C with equal probability($\frac{1}{2}$). The HR says "B will be fired". Does this help A at all?

   [Bonus Question: Can you think of a question for which the same answer by the HR ["B will be fired"] helps A?]

   Solution: Look at the following joint distribution table for your answers.

	HR names B	HR names C
A is retained	$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$	$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$
B is retained	0	$\frac{1}{3}$
C is retained	$\frac{1}{3}$	0

Note the probability of A being retained is $\frac{1}{3}$. Given HR named B the probability of A being retained is $\frac{1/6}{1/6+1/3} = \frac{1}{3}$. Therefore this question doesn't help A.

Clue for Bonus question: modify the question of A to "Will B (or C) be fired?"

7. (a) Let A and B be events with $0 < P(A \cap B) < P(A) < P(B) < P(A \cup B) < 1$.You are hoping that both A and B occurred. Which of the following pieces of information would you be happiest to observe: that A occurred, that B occurred, or that $A \cup B$ occurred?

   (b) Given a random experiment where two standard dices are rolled consecutively, construct two events such that the given condition is satisfied. Verify your claim.

   Solutions: (a) $P(A \cap B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}$ and $P(A \cap B|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$. We see $P(B) > P(A)$ therefore $P(A \cap B|B) < P(A \cap B|A)$. Thus we would be happy to observe A. (Same arguement works with $A \cup B$, replace B with $A \cap B$).

   (b) A : Both die outcomes even number, and, B : Dice-1 outcomes 6.

8. Suppose two suspects X and Y are equally likely to have committed a crime. However, new evidence found a blood type matching X which is present only in 5% of the population.

   (a) Given this information, what is the probability that X has committed the crime?

   (b) What is the probability that Y's blood type matches the one at crime scene?

   Solution:

   (a) Let's denote X and Y to be the events that X and Y are guilty, respectively. Let, B be the event that X's blood type matches the one at the crime scene. By Bayes Rule, we have,

   $$\Pr(X|B) \times \Pr(B) = \Pr(B|X) \times \Pr(X)$$

   We break down $\Pr(B)$ by using the law of total probability for conditional expectations.

   $$\Pr(X|B) = \frac{\Pr(B|X) \times \Pr(X)}{\Pr(B|X) \times \Pr(X) + \Pr(B|Y) \times \Pr(Y)} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{20} \times \frac{1}{2}} = \frac{20}{21}$$

   (b) Let C denote the event that Y's blood type matches the one at the crime scene. Again we use similar breakdown of conditioning on whether Y is guilty or not. The answer should be $\frac{2}{21}$.

9. Suppose there are r red balls and b blue balls in a box. Each time a ball is drawn, k balls of the same colour are added to the box. Let, R_n and B_n be the event that the n-th ball drawn is Red and Blue, respectively. Find $P(R_n)$ and $P(B_n)$.

Solution: We proceed with a proof by induction to show that $P(R_n) = \frac{r}{r+b}$ and $P(B_n) = \frac{b}{r+b}$. Base case obviously true. We assume it is true for $n-1$ and extend it to n.

$$\begin{aligned} P(R_n) &= P(R_n \cap R_{n-1}) + P(R_n \cap B_{n-1}) \\ &= P(R_n|R_{n-1})P(R_{n-1}) + P(R_n|B_{n-1})P(B_{n-1}) \\ &= \frac{r+k-1}{r+b+k-1} \cdot \frac{r}{r+b} + \frac{r}{r+b+k-1} \cdot \frac{b}{r+b} \\ &= \frac{r}{r+b} \end{aligned}$$

10. Consider the random experiment of tossing an unbiased coin twice. In this setup, construct three events X, Y, Z such that:

    (a) If X and Y are independent and Y and Z are independent, but X and Z are dependent.

    (b) X and Y are independent, however they are conditionally dependent given Z.

    (c) X and Y are dependent, however they are conditionally independent given Z.

    Solution:

    (a) X and Z are head and tail in first toss, respectively. Y be the event that second toss come ups head.

    (b) X be head in first toss, Y be two different outcomes in the two tosses and Z be the second toss comes up head.

11. Suppose you are playing an online game where the opponent is equally likely to be an amateur or a pro, giving you a chance of success of 80% and 20%, respectively. You play two matches against the player.

    (a) Given you have won the first match, what is the probability that you will win the second match?

    (b) Given you have won at least one match, what is the probability that you were playing a pro?

    Solution: Bayes Theorem

    (a) Let W_i denote the event that you have won the $i-th$ match. We have to find out $\Pr(W_2|W_1) = \frac{\Pr(W_2 \cap W_1)}{\Pr(W_1)}$.
    We have, $\Pr(W_1) = \frac{4}{5} \times \frac{1}{2} + \frac{1}{5} \times \frac{1}{2} = \frac{1}{2}$. Also, $\Pr(W_1 \cap W_2) = \frac{16}{25} \times \frac{1}{2} + \frac{1}{25} \times \frac{1}{2} = \frac{17}{50}$.
    Then, $\Pr(W_2|W_1) = \frac{\frac{17}{50}}{\frac{1}{2}} = \frac{34}{50}$.

    (b) Similar approach.

12. Let $X_1, X_2, ..., X_n$ be random events occurring with probability $p_1, p_2, ..., p_n$, respectively. Show that $P(\cup_{i=1}^{n} X_i) \leq \Sigma_{i=1}^{n} P(X_i)$.

    Solution: Show that $P(A \cup B) \leq P(A) + P(B)$, and use induction.

13. Consider a two player (A, B) game where each player tosses a coin and A gets 1 point if it comes up head, and B gets 1 point otherwise. The first one to get to 10 points would win the game. However, the game is interrupted when the score of A and B is 8 and 6, respectively. Calculate the probability of A winning the game if:

(a) The coin is unbiased.

(b) The coin comes up head with probability p.

Solution:

(a) Observe that game ends in 5 turns as A and B can get 1 and 3 points without winning,respectively. Also, observe that out of these 5 rounds, B must win ≥ 4 rounds to win the game. Calculate the probabilities.

(b) Replace the simple counting with Binomial Distribution.

14. Suppose you perform a Bernoulli Trial with an unknown probability of success p repeatedly, resulting in n consecutive successes. What would be your estimate of p?

[*Hint: Initially, nothing is known about p. Assume that* $p \sim U(0, 1)$.]

Solution:

$$\begin{aligned} P(X_{n+1} = 1|S_n = n) &= \int_0^1 P(X_{n+1} = 1|S_n = n, p)P(p|S_n = n)dp \\ &= \int_0^1 P(X_{n+1} = 1|p)P(p|S_n = n)dp \\ &= \int_0^1 pP(p|S_n = n)dp \end{aligned}$$

Now

$$\begin{aligned} P(p|S_n = n) &= \frac{P(S_n = n|p)P(p)}{P(S_n = n)} \\ &= \frac{P(S_n = n|p)}{\int_0^1 P(S_n = n|p)P(p)dp} \\ &= \frac{p^n}{\int_0^1 p^n dp} \\ &= (n+1)p^n \end{aligned}$$

where $P(p) = 1$ from pdf of $U(0, 1)$. Therefore,

$$\begin{aligned} P(X_{n+1} = 1|S_n = n) &= \int_0^1 (n+1)p^{n+1}dp \\ &= \frac{n+1}{n+2} \end{aligned}$$

15. Let $\mathcal{A}$ be an algorithm that has a one-sided error and whose failure probability is at most $(1 - 1/{}^nC_2)$. Show how with independent runs of $\mathcal{A}$, the success probability can be amplified.

Solution: Suppose we repeat $\mathcal{A}$ for t many times. As $\mathcal{A}$ has one-sided error (if TRUE $\mathcal{A}$ returns TRUE, if FALSE then $\mathcal{A}$ returns FALSE w/ good probability) we return FALSE if in any of the t repeats $\mathcal{A}$ returns FALSE. Therefore failure probability is if in all the t trials the algorithm $\mathcal{A}$ fails that is

$$\left(1 - \frac{1}{{}^nC_2}\right)^t \leq \frac{1}{e}$$

for $t = n^2$.

Here, we use the inequality $(1 + x) \leq e^x$

16. Suppose you are given a biased coin with unknown probabilities. The goal is to simulate an unbiased coin,i.e. success and fail.

    (a) Think of a basic strategy for the simulation. *[Hint: Try flipping the coin twice.]*

    (b) Given that the biased coin lands on head with probability p, calculate how many number of flips are required in expectation to obtain our outcome?

    (c) Given that the biased coin lands on head with probability $\frac{2}{3}$, can you construct a better(faster) strategy to simulate an unbiased coin?

Solution: (a) In the sample space {HH, TH, HT, TT} the events HT and TH will always have same probability whatever be the bias p. So we keep tossing a coin two times in a single run of the random experiment. If the outcome of both the tosses appears to be the same we discard the experiment immediately. Therefore our new sample space = {HT, TH} both with equal probabilities.

(b) Success prob of an experiment is $P(\text{HT}) + P(\text{TH}) = 2p(1-p)$. Therefore Success of an experiment is a geometric random variable with success probability $2p(1-p)$. Therefore expected number of flips $= \frac{2}{2p(1-p)} = \frac{1}{p(1-p)}$.

(c) The main idea here is to find two equally likely set of disjoint events that gives us a higher probability of succeeding per round. Observe that for $p = \frac{2}{3}$, $P(HH) = \frac{4}{9}, P(HT) = P(TH) = \frac{2}{9}$, and $P(TT) = \frac{1}{9}$. Here, we can choose $\{HH\}$ and $\{HT, TH\}$ as our two equally likely events to get success in $\frac{9}{4}$ flips in expectation instead of $\frac{9}{2}$ flips required by our original strategy.