INDIAN STATISTICAL INSTITUTE

M. Tech (CS) - I Year, 2018-2019 (Semester - I)

Probability and Stochastic Processes

Problem Sheet III

(Q1) A permutation is a one-to-one and onto function $\pi : A \to A$, where $A = \{1, 2, \ldots, n\}$. A fixed-point of a permutation is defined as an index i for which $\pi(i) = i$. A cycle of a permutation is defined as a set of indices $\{i_1, i_2, \ldots, i_k\} \in A$, $k \leq n$, such that $\pi(i_1) = i_2$, $\pi(i_2) = i_3$, …, $\pi(i_{k-1}) = i_k$, and $\pi(i_k) = i_1$.

(i) Design a method that generates any permutation of A with equal probability $\frac{1}{n!}$. Prove your result.

(ii) What is the expected number of fixed points in a random permutation?

(iii) What is the expected number of cycles in a random permutation?

(Q2) Independent trials, each resulting in a success with probability p or a failure with probability $q = 1 - p$, are performed. Compute the probability that a run of n consecutive successes occurs before a run of m consecutive failures. Note that there will be no m consecutive failures before n consecutive successes.

(Q3) You are given that at least one of the events A_r, $1 \leq r \leq n$, is certain to occur, but certainly no more than two occur. If $\Pr(A_r) = p$, and $\Pr(A_r \cap A_s) = q$, $r \neq s$, show that $p \geq \frac{1}{n}$ and $q \leq \frac{2}{n}$.

(Q4) There are n letters marked for n envelopes. The letters are mixed up and put randomly inside the envelopes. A *match* occurs if a letter goes into the envelope it is marked for. What is the probability of exactly k matches?

(Q5) Suppose that X and Y are i.i.d. geometric random variables with parameter p. Show that

$$\Pr(X = i \mid X + Y = n) = \frac{1}{n-1}, \quad i = 1, \ldots, n-1$$

(Q6) Let X and Y be independent variables, X being equally likely to take any value in $\{0, 1, \ldots, m\}$, and Y similarly in $\{0, 1, \ldots, n\}$. Find the mass function of $Z = X + Y$.

(Q7) A communication system consists of n components, each of which will independently function with probability p. The total system will be able to operate effectively if at least one-half of its components function. For what values of p is a 5-component system more likely to operate effectively than a 3-component system?

(Q8) Show that given an undirected graph G with m edges, there is a partition of V into two disjoint sets A and B such that at least $m/2$ edges connect a vertex in A to a vertex in B.