INDIAN STATISTICAL INSTITUTE

End Semestral Examination

M. Tech (CS) - I Year, 2018-2019 (Semester - I)

Probability and Stochastic Processes

Problem Sheet II

(Q1) Independent trials, consisting of rolling a pair of fair dice, are performed. What is the probability that an outcome of 5 appears before an outcome of 7 when the outcome of a roll is the sum of dice?

(Q2) A permutation is a one-to-one and onto function $\pi : A \to A$, where $A = \{1, 2, \ldots, n\}$. A fixed-point of a permutation is defined as an index i for which $\pi(i) = i$. A cycle of a permutation is defined as a set of indices $\{i_1, i_2, \ldots, i_k\} \in A$, $k \leq n$, such that $\pi(i_1) = i_2$, $\pi(i_2) = i_3$, $\ldots$, $\pi(i_{k-1}) = i_k$, and $\pi(i_k) = i_1$.

(i) Design a method that generates any permutation of A with equal probability $\frac{1}{n!}$. Prove your result.

(ii) What is the expected number of fixed points in a random permutation?

(iii) What is the expected number of cycles in a random permutation?

(Q3) If E and F are independent, then so are (i) E and $\overline{F}$; and (ii) $\overline{E}$ and $\overline{F}$.

(Q4) If $\Pr(A) > 0$, show that $\Pr(AB \mid A) \geq \Pr(AB \mid A \cup B)$.

(Q5) Prove that if $E_1, E_2, \ldots, E_n$ are independent events, then

$$\Pr\left(E_1 \cup E_2 \cup \cdots \cup E_n\right) = 1 - \prod_{i=1}^{n} \left(1 - \Pr(E_i)\right)$$

(Q6) Assume that the events A_1, A_2, A_3 and A_4 are independent. Also, $\Pr(A_3 \cap A_4) > 0$. Show that $\Pr(A_1 \cup A_2 \mid A_3 \cap A_4) = \Pr(A_1 \cup A_2)$.

(Q7) Consider a record of n wins and m losses for a team. Assume that all $\frac{(n+m)!}{n!m!}$ orderings of n wins and m losses are equally likely. We define a run as a consecutive sequence of wins. Find the probability that there will be exactly r runs of wins.

(Q8) A parallel system having n components functions when at least one of the components functions. A component i, independent of other components, functions with probability p_i, $i = 1, \ldots, n$. What is the probability that the system functions?

(Q9) Let A, B, and C be independent events, with $\Pr(C) > 0$. Prove that A and B are conditionally independent given C.

(Q10) Let X be a random variable defined over a sample space Ω such that $\mathrm{E}[X] = \mu$. Show that $\Pr(X \geq \mu) > 0$ and $\Pr(X \leq \mu) > 0$.

[Hints: Can you try to prove using contradiction?]