INDIAN STATISTICAL INSTITUTE

<u>Mid Semestral Examination</u>

M. Tech (CS) - I Year, 2019-2020 (Semester - I)

Probability and Stochastic Processes

Date: 09.09.2019 | Maximum Marks: 60 | Duration: 2.5 Hours

Note:

Answer all questions in Group-A and answer as much as you can in Group-B but the maximum you can score in Group-B is 36.

$E[X]$ and var$[X]$ denote the expectation and variance of the random variable X, respectively.

Group A

(QA1) (a) Assuming that all conditioning events have positive probability, prove that

$$\Pr\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} \Pr\left(A_i \mid \bigcap_{j=1}^{i-1} A_j\right)$$

.

(Ans:) This is the chain multiplication rule and was done in the class.

(b) Let the probabilities of events A and B be Pr(A) and Pr(B), respectively and the probability of event $A \cap B$ be Pr$(A \cap B)$. Find out the probability that exactly one of the events A and B will occur.

(Ans:) The probability is $\Pr\left((A \cap \overline{B}) \cup (\overline{A} \cap B)\right) = \Pr(A) + \Pr(B) - 2\Pr(A \cap B)$.

[4+4=8]

(QA2) Two gamblers, G_1 and G_2 bet on the outcomes of successive flips of a coin. On each flip, if the coin comes up heads, G_1 collects Rs. 1 from G_2; and if the coin comes up tails, G_2 collects Rs. 1 from G_1. The gamblers continue to do this until one of them runs out of money. Assume that the successive flips of coins are independent and the probability of a coin flip resulting in a head is p, and resulting in a tail is $1-p$. Show that with probability 1, either G_1 or G_2 will wind up with all the money. [8]

(Ans:) This is the gamblers' ruin problem and was done in the class.

(QA3) (a) For events A_i, $i = 1, \ldots, n$, show that $\Pr\left(\bigcap_{i=1}^{n} A_i\right) \geq 1 - n + \sum_{i=1}^{n} \Pr(A_i)$

(Ans:) This is the generalized Bonferroni's inequality and was done in the class.

(b) Let $C_1, \ldots, C_n$ be disjoint events that form a partition of the state space. Let also A and B be events such that $\Pr(B \cap C_i) > 0$ for all i. Show that

$$\Pr(A \mid B) = \sum_{i=1}^{n} \Pr(C_i \mid B) \Pr(A \mid B \cap C_i).$$

(Ans:) We have for disjoint C_i's

$$\Pr(A \cap B) = \sum_{i=1}^{n} \Pr((A \cap B) \cap C_i).$$

Use the multiplication rule,

$$\Pr((A \cap B) \cap C_i) = \Pr(B) \Pr(C_i|B) \Pr(A|B \cap C_i)$$

Combine the above two; divide by $\Pr(B)$ and use the formula $\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$, to obtain the desired result. [3+5=8]

Group B

(QB1) (a) Let A and B be events with $\Pr(A) = \frac{3}{4}$ and $\Pr(B) = \frac{1}{3}$. Show that $\frac{1}{12} \leq \Pr(A \cap B) \leq \frac{1}{3}$.

(Ans:) $\Pr(A \cap B) \leq \Pr(A)$ and $\Pr(A \cap B) \leq \Pr(B)$. So, $\Pr(A \cap B) \leq \min\{\Pr(A), \Pr(B)\} = \frac{1}{3}$.
For the other side, $\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B) \geq \frac{3}{4} + \frac{1}{3} - 1 = \frac{1}{12}$.

(b) A fair coin is tossed repeatedly. Show that, with probability 1, a head turns up sooner or later.

(Ans:) $\Pr($ a head turns up sooner or later$) = 1 - \Pr($ no head ever turns up$)$. As we want $\Pr($ a head turns up sooner or later$) = 1$, we will want to show

$$\Pr(\text{ no head ever turns up}) = 0$$

.

$$\begin{aligned} \Pr(\text{ no head ever turns up}) &= \lim_{n \to \infty} \Pr(\text{ no head turns up in first } n \text{ tosses}) \\ &= \lim_{n \to \infty} \left(\frac{1}{2}\right)^n \\ &= 0 \end{aligned}$$

[4+5=9]

(QB2) Let A and B be events with $\Pr(A) > 0$ and $\Pr(B) > 0$. We say that an event B *suggests* an event A, if $\Pr(A \mid B) > \Pr(A)$, and *does not suggest* event A if $\Pr(A \mid B) < \Pr(A)$. Show that

- B suggests A if and only if A suggests B.

(Ans:) We first prove B suggests $A \Rightarrow A$ suggests B.

$$
\begin{aligned}
\Pr(A \mid B) &> \Pr(A) \\
\frac{\Pr(A \cap B)}{\Pr(B)} &> \Pr(A) \\
\Pr(A \cap B) &> \Pr(A) \cdot \Pr(B) \\
\frac{\Pr(A \cap B)}{\Pr(A)} &> \Pr(B) \\
\Pr(B \mid A) &> \Pr(B) \Rightarrow A \text{ suggests } B
\end{aligned}
$$

By symmetry, the other direction follows.

- Assume that $\Pr(\overline{B}) > 0$. Show that B suggests A if and only if $\overline{B}$ does not suggest A.

(Ans:) $\Pr(A) = \left(\Pr\left(\overline{B}\right) + \Pr(B)\right)\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid \overline{B})\Pr(\overline{B})$. So, we have

$$
\begin{aligned}
\left(\Pr\left(\overline{B}\right) + \Pr(B)\right)\Pr(A) &= \Pr(A \mid B)\Pr(B) + \Pr(A \mid \overline{B})\Pr(\overline{B}) \\
\Pr\left(\overline{B}\right)\left(\Pr(A) - \Pr(A \mid \overline{B})\right) &= \Pr(B)\left(\Pr(A \mid B) - \Pr(A)\right)
\end{aligned}
$$

As both $\Pr(B)$ and $\Pr(\overline{B}) > 0$, the result follows.

[4+5=9]

(QB3) (a) There are n people who keep their umbrellas in a box and then pick an umbrella at random. Find the variance of the number of people who pick their own umbrella.

(Ans:) Let X_i be the r.v. denoting that the i-th person has own umbrella, i.e.,

$$
X_i = \begin{cases} 1 & \text{if } i\text{-th person gets the } i\text{-th umbrella;} \\ 0 & \text{o/w.} \end{cases}
$$

and let $X = \sum_{i=1}^{n} X_i$. We need to compute var$[X]$.

$$
\text{var}[X] = \sum_{i=1}^{n} \text{var}[X_i] + \sum_{1 \le i \ne j \le n} \text{cov}[X_i X_j]
$$

Each X_i is a Bernoulli with parameter $p = \frac{1}{n}$. So, $E[X_i] = \frac{1}{n}$ and var$[X_i] = \frac{1}{n}\left(1 - \frac{1}{n}\right)$.

$$
\text{cov}[X_i X_j] = E[X_i\, X_j] - E[X_i]\, E[X_j]
$$

$$
\begin{aligned}
\mathrm{E}[X_i X_j] &= \Pr(X_i = 1 \text{ and } X_j = 1) \\
&= \Pr(X_i = 1) \cdot \Pr(X_j = 1 \mid X_i = 1) \\
&= \frac{1}{n} \cdot \frac{1}{n-1}
\end{aligned}
$$

So, we have

$$
\begin{aligned}
\mathrm{cov}[X_i X_j] &= E[X_i\, X_j] - E[X_i]\, E[X_j] \\
&= \frac{1}{n} \cdot \frac{1}{n-1} - \frac{1}{n} \cdot \frac{1}{n} \\
&= \frac{1}{n^2(n-1)}
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
\mathrm{var}[X] &= \sum_{i=1}^{n} \mathrm{var}[X_i] + \sum_{1 \le i < j \le n} \mathrm{cov}[X_i X_j] \\
&= \sum_{i=1}^{n} \frac{1}{n}\left(1 - \frac{1}{n}\right) + \sum_{1 \le i \ne j \le n} \frac{1}{n^2(n-1)} \\
&= \left(1 - \frac{1}{n}\right) + n(n-1)\frac{1}{n^2(n-1)} \\
&= 1
\end{aligned}
$$

(b) Let $X_1, \ldots, X_n$ be i.i.d. geometric random variables, each with success probability p. Prove that $\Pr(\max_i X_i > t) \le n(1-p)^t$.

(Ans:) X_i's are i.i.d. geometric random variables. So, we have

$$
\begin{aligned}
\Pr(X_i > t) &= 1 - \Pr(X_i \le t) \\
&= 1 - \left(1 - (1-p)^t\right) \\
&= (1-p)^t
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
\Pr(\max_i X_i > t) &= \Pr(\text{ at least one } X_i > t) \\
&= \Pr\left(\bigcup_{i=1}^{n} (X_i > t)\right) \\
&\le \sum_{i=1}^{n} \Pr(X_i > t) \text{ (using union bound)} \\
&= n(1-p)^t
\end{aligned}
$$

[4+5=9]

(QB4) Let $X_1, \ldots, X_n$ be independent random variables such that $E[X_i] \neq 0$. Show that

$$\frac{\text{var}\left(\prod_{i=1}^{n} X_i\right)}{\prod_{i=1}^{n} E[X_i]^2} = \prod_{i=1}^{n}\left(\frac{\text{var}(X_i)}{E[X_i]^2} + 1\right) - 1$$

[9]

(Ans:) Use independence and the formula of variance, $\text{var}[X] = E[X^2] + E[X]^2$, to get

$$\begin{aligned}
\text{var}\left(\prod_{i=1}^{n} X_i\right) &= E\left[\left(\prod_{i=1}^{n} X_i\right)^2\right] - \left(E\left[\prod_{i=1}^{n} X_i\right]\right)^2 \\
&= E\left[\prod_{i=1}^{n} X_i^2\right] - \left(\prod_{i=1}^{n} E[X_i]\right)^2 \\
&= \prod_{i=1}^{n} (E[X_i^2]) - \prod_{i=1}^{n} (E[X_i])^2 \\
&= \prod_{i=1}^{n} \left(\text{var}[X_i] + E[X_i]^2\right) - \prod_{i=1}^{n} (E[X_i])^2
\end{aligned}$$

Now, divide both sides by $\prod_{i=1}^{n} E[X_i]^2$ to get the result.

(QB5) A *tournament* on a set V of n players is an orientation $T = (V, E)$ of the edges of the complete graph on the set of vertices V. Thus for every two distinct elements x and y of V, either (x, y) or (y, x) belongs to E, but not both. A simple interpretation of *tournament* is in terms of games where each distinct pair x, y of players, $x, y \in V$, play a single match; the outcome of the games are either win or loss. (x, y) is in the *tournament* if and only if x beats y.

T has the property S_k if for every set of k players there is one who beats them all.

Show that if $\binom{n}{k}(1 - 2^{-k})^{n-k} < 1$, then there is a tournament on n vertices that has the property S_k. [9]

[**Hints:** Can you use probabilistic methods? You can consider a random tournament on V by choosing either edge (i, j) or (j, i), where each of these two choices is equally likely. Consider a fixed subset K of V, $|K| = k$ and let $\mathcal{E}_K$ denote the event that there is no vertex that beats all the members of K. What is $\Pr(\mathcal{E}_K)$?]

(Ans:) Consider a fixed subset K (($K \subset V$) of size k from a random tournament $V = \{1, 2, \ldots, n\}$.

Notice that for each vertex $v \in V \setminus K$, the probability that v does not beat all the members of K is $1 - 2^{-k}$ and all of these $n - k$ events for a choice of v are independent. Let $\mathcal{E}_K$ be the event that there is no player (vertex) who beats all the members of K. So, $\Pr(\mathcal{E}_K) = (1-2^{-k})^{n-k}$. We next use the union bound and the condition given in the question to deduce the result, as follows:

$$\Pr\left(\bigcup_{K\subset V,|K|=k} \mathcal{E}_K\right) = \sum_{K\subset V,|K|=k} \Pr\left(\mathcal{E}_K\right) = \binom{n}{k}\left(1-2^{-k}\right)^{n-k} < 1.$$