INDIAN STATISTICAL INSTITUTE

Class Test II

M Tech (CS) – I Year, 2019-2020 (Semester - I)

Probability and Stochastic Processes

Date: 15.11.2019 | Maximum Marks: 30 | Duration : 1.0 hour

Note: The question paper is of 40 marks. Answer as much as you can, but the maximum you can score is 30. Answer a question within its allotted box.

Course: (M Tech/JRF/PLP) ______________________

Name: ______________________ **Roll Number:** ______________________

(Q1) A sequence X_n of random variables is said to converge to a number c in the mean square, if $\lim\limits_{n\to\infty} E[(X_n - c)^2] = 0$. Show that convergence in the mean square implies convergence in probability. [10]

Let X_n converges to c in the mean square. Using the Markov inequality, we get

$$\Pr(|X_n - c| \geq \epsilon) = \Pr(|X_n - c|^2 \geq \epsilon^2) \leq \frac{E\left[(X_n - c)^2\right]}{\epsilon^2}.$$

This implies $\lim_{n\to\infty} \Pr(|X_n - c| \geq \epsilon) = 0$, which is the desired result. ◀

(Q2) Jobs are processed, one at a time. The processing times of jobs are independent random variables, uniformly distributed in $[1, 10]$. Find or approximate the probability that the number of jobs processed within 200 time units is at least 75. [10]

Let X_i be the random variable that denotes the processing time of the i^{th} job, $i \in \mathbb{N}$. Note that $X_i \in [1, 10]$ and, is uniformly distributed. $Z = \frac{X_1+\ldots+X_{75}-75\cdot\mu}{\sqrt{75}\sigma}$, where μ and σ are the mean and standard deviation of each X_i. So, $\mu = 11/2$ and $\sigma \approx 2.59$.

Observe that we have to compute the following.

$$\begin{aligned} & \Pr(X_1 + \ldots + X_{75} \leq 200) \\ \approx \ & \Pr\left(Z \leq -9.47\right) \\ \approx \ & \Phi(-9.47) \text{ (by CLT)} \end{aligned}$$

◀

(Q3) X and Y are independent r.v.'s, distributed normally, as $N(0,1)$. Show that, for any fixed θ, the random variables $U = X\cos\theta + Y\sin\theta$ and $V = -X\sin\theta + Y\cos\theta$ are independent and find their distributions. [10]

The MGF of X, i.e., $M_X(t) = e^{t^2/2}$. Also, $M_Y(t) = e^{t^2/2}$. Using the fact that $M_{aX+bY}(t) = M_X(at)\cdot M_Y(bt)$, the MGF of U is

$$\begin{aligned} M_U(t) &= M_X(t\cos\theta)\cdot M_Y(t\sin\theta) \\ &= e^{t^2\cos^2\theta/2}\cdot e^{t^2\sin^2\theta/2} = e^{t^2/2}. \end{aligned}$$

This implies that U follows $N(0,1)$.
Similarly, it can be shown that V follows $N(0,1)$.
Now, we have to show that U and V are independent. First, observe that U and V are obtained by rotating X and Y axis by an angle of θ. So, X and Y can be obtained from rotating U and V line by an angle of $-\theta$. Hence, $U \le u, V \le v$ is equivalent to $X \le u\cos\theta - v\sin\theta, Y \le u\sin\theta + v\cos\theta$.

$$\begin{aligned} & \Pr(U \le u, V \le v) \\ = & \Pr(X \le u\cos\theta - v\sin\theta, Y \le u\sin\theta + v\cos\theta) \\ = & \Pr(X \le u\cos\theta - v\sin\theta)\cdot\Pr(Y \le u\sin\theta + v\cos\theta)\ (\because\ X \text{ and } Y \text{ are independent.}) \\ = & \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u\cos\theta - v\sin\theta} e^{-x^2/2}\cdot\int_{-\infty}^{u\sin\theta+v\cos\theta} e^{-y^2/2}dy \\ = & \frac{1}{2\pi}\int_{-\infty}^{u\cos\theta - v\sin\theta} e^{-x^2/2}\cdot\int_{-\infty}^{u\sin\theta+v\cos\theta} e^{-y^2/2}dy \\ = & \frac{1}{2\pi}\int_{-\infty}^{u\cos\theta - v\sin\theta}\int_{-\infty}^{u\sin\theta+v\cos\theta} e^{-(x^2+y^2)/2}dxdy \\ = & \frac{1}{2\pi}\int_{-\infty}^{u}\int_{-\infty}^{v} e^{-(u^2+v^2)/2}dudv \text{ (using } \textit{Jacobian}\text{)} \\ = & \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} e^{-u^2/2}du\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{v} e^{-v^2/2}dv \\ = & \Pr(U \le u)\cdot\Pr(V \le v)\ (\because\ U \text{ and } V \text{ follow } N(0,1)) \end{aligned}$$

This implies U and V are independent. ◀

(Q4) (i) Prove that, in a Markov chain, if one state in a communicating class is recurrent, then all states in that class are recurrent. (ii) Find a simple expression for $P^t_{0,0}$ for the two state (i.e. $\{0,1\}$) Markov chain with the following transition matrix $\mathbf{P}$

$$\begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

[5+5=10]

The Markov chain can come back to state 0 after t steps either from state 0 (with a probability of p) or from state 1 (with a probability of $1-p$). In the last case, the Markov chain must have reached state 1 from state 0 in $t-1$ steps. So, we can write

$$\begin{aligned} P^t_{0,0} &= P^{t-1}_{0,0} \cdot p + P^{t-1}_{0,1} \cdot (1-p) \\ &= P^{t-1}_{0,0} \cdot p + (1 - P^{t-1}_{0,0}) \cdot (1-p) \quad \{ \text{as } P^{t-1}_{0,0} + P^{t-1}_{0,0} = 1\} \\ &= (1-p) + P^{t-1}_{0,0}(2p-1) \end{aligned}$$

This is a linear recurrence. Solve this to get the value of $P^t_{0,0}$. ◄