INDIAN STATISTICAL INSTITUTE

Class Test II

M Tech (CS) – I Year, 2019-2020 (Semester - I)

Probability and Stochastic Processes

Date: 30.10.2019 Maximum Marks: 30 Duration : 1.0 hour

Note: The question paper is of 40 marks. Answer as much as you can, but the maximum you can score is 30. Answer a question within its allotted box.

Course: (M Tech/JRF/PLP) ____________________

Name: ____________________ **Roll Number:** ____________________

(Q1) Show that the continuous random variables X and $-X$ have the same distribution function if and only if $f_X(x) = f_X(-x)\ \forall x \in \mathbb{R}$. [10]

Let $Y = aX$. Then, the CDF of Y is related to the CDF of X as:

$$F_Y(y) = \Pr(Y \leq y) = \Pr(aX \leq y) = \Pr(X \leq \frac{y}{a} = F_X(\frac{y}{a})).$$

Differentiate to get the relations between PDF as, $f_Y(y) = \frac{1}{a} f_X(\frac{y}{a})$. If $a = -1$, we get $f_Y(y) = -f_X(-y)$.

Now, let us try to deduce the relation between f_X and f_{-X}. We will apply the usual technique of getting the CDF and then differentiating to get the PDF.

First, we show X and $-X$ having the same distribution, i.e., $f_X(x) = f_{-X}(x)$ implies $f_X(x) = f_X(-x)\ \forall x \in \mathbb{R}$. As they have same distribution, they have the same CDF, and hence

$$F_X(x) = F_{-X}(x) = \Pr(-X \leq x) = \Pr(X \geq -x) = 1 - \Pr(X \leq -x) = 1 - F_X(-x).$$

Differentiating, and using the deduced fact that $f_Y(y) = -f_X(-y)$, we get the desired implication that $f_X(x) = f_X(-x)$.

Conversely, we need to prove $f_X(x) = f_X(-x)\ \forall x \in \mathbb{R}$ implies $f_X(x) = f_{-X}(x)$, or equivalently, $F_X(x) = F_{-X}(x)$.

$$F_{-X}(x) = \Pr(-X \leq x) = \Pr(X \geq -x) = \int_{-x}^{\infty} f_X(u)\, du = \int_{-x}^{\infty} f_X(-u)\, du.$$

Let $-u = v$. So, we have

$$\int_{-x}^{\infty} f_X(-u)\, du = \int_{x}^{-\infty} -f_X(v)\, dv = \int_{-\infty}^{x} f_X(v)\, dv = F_X(x)$$

(Q2) Let $X_1, \ldots, X_n$ be independent uniform random variables over $[0, 1]$. Let $Y = \min(X_1, \ldots, X_n)$. Show that $\mathrm{E}[Y] = \frac{1}{n+1}$. [10]

We know for a continuous random variable X, that takes only nonnegative values, and has a density function $f(x)$,

$$\begin{aligned}
\int_{x=0}^{\infty} \Pr\left(X \geq x\right)\, dx &= \int_{x=0}^{\infty} \int_{y=x}^{\infty} f(y)\, dy\, dx \\
&= \int_{y=0}^{\infty} \int_{x=0}^{y} f(y)\, dx\, dy \\
&= \int_{y=0}^{\infty} y f(y)\, dy \\
&= E[X].
\end{aligned}$$

Now, as per the question, $Y_1 = \min(X_1, \ldots, X_n)$ and Y_1 takes value in the range $[0, 1]$. We want to find $E[Y_1]$ which is $\int_{y=0}^{1} \Pr(Y_1 \geq y)$. So, we need to figure out the probability of $\Pr(Y_1 \geq y)$.

$$\begin{aligned}
\Pr(Y_1 \geq y) &= \Pr(\min(X_1, \ldots, X_n) \geq y) \\
&= \Pr\left((X_1 \geq y) \cap (X_2 \geq y) \cap \cdots \cap (X_n \geq y)\right) \\
&= \prod_{i=1}^{n} \Pr(X_i \geq y) \\
&= (1 - y)^n
\end{aligned}$$

Therefore, $E[Y_1] = \int_{y=0}^{1} (1 - y)^n \, dy = \frac{1}{n+1}$. ◀

(Q3) Let X be a non-negative random variable with mean μ and continuous distribution function F. Show that $\int\limits_{-\infty}^{a} F(x)dx = \int\limits_{a}^{\infty}[1-F(x)]dx$, if and only if $a = \mu$. [10]

For a nonnegative random variable X, we know that

$$E[X] = \int_0^\infty \Pr(X > x)\, dx$$

The continuous distribution function $F(x) = \Pr(X \leq x) = 1 - \Pr(X > x)$. So, $\Pr(X > x) = 1 - F(x)$. Using this result in the above equality, we have

$$\begin{aligned} E[X] &= \int_0^\infty \Pr(X > x)\, dx \\ \text{or, } \mu &= \int_0^\infty [1 - F(x)]\, dx \\ \text{or, } \mu &= \int_0^\mu [1 - F(x)]\, dx + \int_\mu^\infty [1 - F(x)]\, dx \\ \text{or, } \int_\mu^\infty [1 - F(x)]\, dx &= \mu - \int_0^\mu [1 - F(x)]\, dx \end{aligned}$$

Also, note that

$$\begin{aligned} \mu &= \int_0^\mu F(x)\, dx + \int_0^\mu (1 - F(x))\, dx \\ \text{or, } \mu - \int_0^\mu (1 - F(x))\, dx &= \int_0^\mu F(x)\, dx \end{aligned}$$

From the above two inequalities, we have

$$\int_0^\mu F(x)\, dx = \int_\mu^\infty [1 - F(x)]\, dx$$

With $a = \mu$, we have

$$\int_\mu^\infty [1 - F(x)]\, dx = \int_0^\mu F(x)\, dx$$

For the proof in the other direction, consider the given statement

$$\int_a^\infty [1 - F(x)]\, dx = \int_0^a F(x)\, dx$$

$F(x)$ is a monotonic increasing function and $1 - F(x)$ is a monotonic decreasing function. They would intersect at a point and their value being same, that point has to be μ. ◀

(Q4) If X is a random variable with mean 0 and finite variance σ^2, then for any $a > 0$, show that $\Pr(X \geq a) \leq \frac{\sigma^2}{\sigma^2+a^2}$. [10]

(Ans:) The proof technique is almost same as the deduction we did for Chebyshev's inequality in the class. Let $b > 0$.

$$\begin{aligned}
\Pr(X \geq a) &= \Pr(X + b \geq a + b) \\
&\leq \Pr\left((X+b)^2 \geq (a+b)^2\right) \\
&\leq \frac{E[(X+b)^2]}{(a+b)^2} \\
&= \frac{\sigma^2 + b^2}{(a+b)^2}
\end{aligned}$$

as $E[(X+b)^2] = E[X^2 + 2b \cdot X + b^2] = E[X^2] + 2b \cdot E[X] + b^2 = \sigma^2 + b^2$. Minimizing the above expression, we have $b = \frac{\sigma^2}{a}$ and putting back into the above Equation, we get the final result. ◀