**INDIAN STATISTICAL INSTITUTE**

**Class Test I**

M Tech (CS) – I Year, 2018-2019 (Semester - I)

*Probability and Stochastic Processes*

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Date: 04.09.2019 Maximum Marks: 30 Duration : 1.0 hour

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**Note:** The question paper is of 40 marks. Answer as much as you can, but the maximum you can score is 30. Answer a question within its allotted box.

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**Course:** (M Tech/JRF/PLP) ____________________

**Name:** ____________________ **Roll Number:** ____________________

(Q1) Let there be $n$ husband-wife couples. The $2n$ persons are arranged into $n$ new pairs. Find the probability that

**(a)** the new pairs will be the original husband-wife pair.

**(b)** that all men are paired with women (not necessarily husband-wife).

[5+5=10]

(Ans:) **(Ans a:)** The possible number of arrangements is $(2n)!$. Of them, fix a couple. There are $n$ of them. They can be permuted in $n!$ ways. Now, in an arrangement, each couple can be arranged in 2 ways, giving $2^n$ for $n$ couples. So, the said probability is $\frac{2^n n!}{(2n)!}$.

**(Ans b:)** Now permute $n$ men in $n!$ ways and $n$ women in $n!$ ways and pair them up in $(n!)^2$ ways. For each such arrangement, one can again order the men and women in 2 ways, leading to $2^n$ for $n$ couples. Thus, the said probability is $\frac{2^n (n!)^2}{(2n)!}$ which for a better form is $\frac{2^n}{\binom{2n}{n}}$. ◄

(Q2) Airlines find that each passenger who reserves a seat fails to turn up with probability $\frac{1}{10}$ independently of the other passengers. So, Indigo Airlines always sells 10 tickets for their 9 seater aeroplane while Air India always sells 20 tickets for their 18 seater aeroplane. Which is more often over-booked? [10]

(Ans:) Let $X$ and $Y$ denote the number of people to whom Indigo and Air India sell tickets, respectively. So, $X$ and $Y$ take value in $[1, \ldots, 10]$ and $[1, \ldots, 20]$. The failure probability of a passenger turning up is $\frac{1}{10}$ and the success probability is $\frac{9}{10}$ for both airlines.

So, $\Pr(X = k) = \binom{10}{k} \left(\frac{9}{10}\right)^k \left(\frac{1}{10}\right)^{10-k}$ and $\Pr(Y = k) = \binom{20}{k} \left(\frac{9}{10}\right)^k \left(\frac{1}{10}\right)^{20-k}$.

Now, the probability of Indigo to be overbooked is

$$\begin{aligned}
\Pr(X > 9) &= \Pr(X = 10) \\
&= \binom{10}{10} \left(\frac{9}{10}\right)^{10} \left(\frac{1}{10}\right)^{10-10} \\
&= \left(\frac{9}{10}\right)^{10} \\
&= a \quad \text{(say)}
\end{aligned}$$

Now, the probability of Air India to be overbooked is

$$\begin{aligned}
\Pr(Y > 18) &= \Pr(Y = 19) + \Pr(Y = 20) \\
&= \binom{20}{19} \left(\frac{9}{10}\right)^{19} \left(\frac{1}{10}\right) + \binom{20}{20} \left(\frac{9}{10}\right)^{20} \\
&= 2\left(\frac{9}{10}\right)^{19} + \left(\frac{9}{10}\right)^{20} \\
&= \left(\frac{9}{10}\right)^{10} \left(2\left(\frac{9}{10}\right)^{9} + \left(\frac{9}{10}\right)^{10}\right) \\
&= a\left(\frac{20}{9}a + a\right) \\
&= a\left(\frac{29}{9}a\right) \\
&> a
\end{aligned}$$

As $\Pr(Y > 18) > \Pr(X > 9)$, Air India is more often overbooked. ◀

(Q3) Let $X$ and $Y$ be independent Bernoulli random variables with parameter $\frac{1}{2}$. Show that $X+Y$ and $|X-Y|$ are dependent though uncorrelated. [10]

(Ans:) Let us look at the covariance of the random variables $X+Y$, and $|X-Y|$.

$$\text{cov}\,(X+Y, |X-Y|) = E[(X+Y)(|X-Y|)] - E[X+Y]\cdot E[|X-Y|]$$

The values of the random variables and their corresponding probabilities are as follows:

| $(X+Y)\,\lvert X-Y\rvert$ | 0 | 1 | 2 |
|---|---|---|---|
| Probability | $\frac{5}{8}$ | $\frac{1}{4}$ | $\frac{1}{8}$ |

| $(X+Y)$ | 0 | 1 | 2 |
|---|---|---|---|
| Probability | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{1}{4}$ |

| $(X-Y)$ | 0 | 1 |
|---|---|---|
| Probability | $\frac{1}{2}$ | $\frac{1}{2}$ |

$$\begin{aligned}\text{cov}\,(X+Y, |X-Y|) &= E[(X+Y)(|X-Y|)] - E[X+Y]\cdot E[|X-Y|] \\ &= \frac{1}{4}+\frac{1}{4}-1\cdot\frac{1}{2} \\ &= 0\end{aligned}$$

So, $X+Y$ and $|X-Y|$ are uncorrelated.

Looking at the probabilities $\Pr(X+Y=0, |X-Y|=0)$, $\Pr(X+Y=0)$ and $\Pr(|X-Y|=0)$, we see that $\Pr(X+Y=0, |X-Y|=0) \neq \Pr(X+Y=0)\cdot\Pr(|X-Y|=0)$. So, they are dependent. ◀

(Q4) There are $n$ urns of which the $i$-th urn contains $i-1$ red balls and $n-i$ blue balls. You pick an urn at random and remove two balls at random without replacement. Find the probability that (a) the second ball is blue; (b) the second ball is blue given the first ball is blue. [5+5=10]

(Ans:) Let $C_i$ denote the color of the $i$-th ball picked. Each urn contains $n-1$ balls, so there are $n(n-1)$ balls in all, of which $\frac{n(n-1)}{2}$ are blue ($B$) and $\frac{n(n-1)}{2}$ are red ($R$).

(a) The answer is $\frac{1}{2}$. You can see it using total probability.

(b) We seek $\Pr(C_2 = B \mid C_1 = B)$ by conditioning on the choice of the urn. $\Pr(C_1 = B) = \frac{\frac{n(n-1)}{2}}{n(n-1)} = \frac{1}{2}$.

$$\Pr(C_2 = B \mid C_1 = B) = \frac{\Pr(C_1, C_2 = B)}{\Pr(C_1 = B)} = \frac{\Pr(C_1, C_2 = B)}{\frac{1}{2}}. \tag{1}$$

Now, to find $\Pr(C_1, C_2 = B)$, we condition on the choice of the urn as

$$\begin{aligned}\Pr(C_1, C_2 = B) &= \sum_{i=1}^{n} \Pr(C_1, C_2 = B \mid i\text{-th urn was chosen}) \cdot \Pr(i\text{-th urn was chosen}) \\ &= \frac{1}{n}\sum_{i=1}^{n} \frac{n-i}{n-1} \cdot \frac{n-i-1}{n-2} \quad [\text{as Pr(i-th urn was chosen)} = \frac{1}{n}]\end{aligned}$$

Replace the above in Equation (2), to get

$$\Pr(C_2 = B \mid C_1 = B) = \frac{\Pr(C_1, C_2 = B)}{\Pr(C_1 = B)} = \frac{\left(\sum_{i=1}^{n} \frac{(n-i)(n-i-1)}{n(n-1)(n-2)}\right)}{\frac{1}{2}} = \frac{\frac{2n(n-1)(n-2)}{6n(n-1)(n-2)}}{\frac{1}{2}} = \frac{2}{3}$$

◀