INDIAN STATISTICAL INSTITUTE

End Semestral Examination

M. Tech (CS) - I Year, 2018-2019 (Semester - I)

Probability and Stochastic Processes

Problem Sheet IV

(Q1) Let $X_1, X_2, \ldots$ be a sequence of independent random variables that are uniformly distributed in $[0, 1]$, and let $Y_n = \min\{X_1, \ldots, X_n\}$. Show that Y_n converges to 0, with probability 1.

(Q2) Consider a sequence X_n of non-negative random variables and suppose that

$$\mathbb{E}\left[\sum_{n=1}^{\infty} X_n\right] \quad < \quad \infty.$$

Show that X_n converges to 0, with probability 1, i.e. almost surely.

(Q3) A machine processes parts, one at a time. The processing times of different parts are independent random variables, uniformly distributed on [1, 5]. Compute an approximate probability that the number of parts processed within 320 time units is at least 100.

(Q4) (a) Can you generate a 2 SAT formula with three clauses that is not satisfiable?

(b) Let $\mathcal{F}$ be a 2 SAT formula with m clauses and n variables. Prove that there exists an assignment to the variables such that at least $\frac{3m}{4}$ clauses will satisfy?

(c) Try to generalize the above for d-SAT.

(Q5) A directed graph $G(V, E)$ is a *tournament* if exactly one out of (x, y) and (y, x) is in E. Prove that there exists a tournament on G having $\frac{n!}{2^{n-1}}$ hamiltonian paths.

(Q6) Let m balls are put into n bins such that the bin for each ball is chosen independently and uniformly at random. Let $N_i, i \in [n]$, be the number of balls in the i^{th} bin. Prove that

(a) $\Pr(\left|N_i - \frac{m}{n}\right| \geq \sqrt{\frac{3m}{n}\ln n}) \leq \frac{2}{n}$.

(b) $\Pr(|N_i - N_j| \geq 2\sqrt{\frac{3m}{n}\ln n}) \leq \frac{4}{n}$.

(Q7) Let there are n persons $P_1, \ldots, P_n$ and n seats $S_1, \ldots, S_n$ such that the seat S_i is assigned to P_i. $P_i's$ come in the increasing order of their indices to occupy seats in the following manner. P_1 occupy one of n seats uniformly at random. Other persons occupy the assigned seat if it is available; otherwise they occupy one of the available seats uniformly at random. Compute the probability that P_n occupies S_n.

(Q8) Let $S \subseteq \{1, \ldots, n\}$ be such that $|S| \leq k \leq n$; $R_1, \ldots, R_{\alpha 2^k}$ be random subsets of $\{1, \ldots, n\}$. Prove that $\Pr(S \subseteq R_i) \geq \frac{1}{2^k}$ and $\Pr(\exists\, i \in [\alpha 2^k] : S \subseteq R_i) \geq 1 - \frac{1}{e^\alpha}$.

(Q9) Let X follows Bin(n, p). Then what is the most probable value X can take. Justify your answer.

(Q10) Let X_0, X_1, ... be a sequence for random variables evolving according to the following rule: $X_0 = 0$, and for all $i \geq 0$

$$P(X_{i+1} = X_i + 1) = \frac{1}{2^{X_i}}, \quad \text{and} \quad P(X_{i+1} = X_i) = 1 - \frac{1}{2^{X_i}}.$$

Prove that $\mathbb{E}\left[2^{X_n}\right] = n + 1$. (Hint: You may use induction and conditional expectation.)