INDIAN STATISTICAL INSTITUTE

Problem Sheet-3

M. Tech (CS) - I Year, 2018-2019 (Semester - I)

Course: *Probability and Stochastic Processes*

(Q1) If you select k elements uniformly at random from $\{1, \ldots, n\}$ without replacement, compute the probability that you do not have two consecutive elements.

(Q2) If you take 3 points uniformly at random from a unit circle then compute the probability that all the three points will lie on a half circle. Can you generalize for n points?

(Q3) Prove that $H(X) \leq \log n$, where X is a random variable that takes value from $\{1, 2, \ldots, n\}$ and $H(X) = -\sum_{i=1}^{n} \mathbb{P}(X = i) \log(\mathbb{P}(X = i))$.

(Q4) Let $f : [n] \to [n]$ be a bijective function and $\pi : [n] \to [n]$ be a random permutation. Compute $\mathbb{E}[X]$ and $\mathbb{V}[X]$, where $X(\pi) := |\{i : f(i) = \pi(i)\}|$.

(Q5) Show that *Markov's inequality* and *Chebyshev's inequality* are tight.

(Q6) There are n rabbits in a forest and m hunters trying to hunt them. Each hunter chooses one rabbit uniformly from the n rabbits and the probability that the i-th hunter can kill its choosen is p. In expectation, how many rabbits will the hunters kill?

(Q7) Suppose we select mn numbers (X_{ij}), where $1 \leq i \leq m$ and $1 \leq j \leq n$, from the interval $[0, 1]$ uniformly at random, and let $X_i = \min_{1 \leq j \leq n} X_{ij}$ where $1 \leq i \leq m$.

(a) Prove that $\mathbb{E}[X_i] = \frac{1}{n+1}$ and $\mathbb{V}[X_i] \leq \frac{1}{(n+1)^2}$.

(b) Prove that $\mathbb{P}\left(|Y - \frac{1}{n+1}| \geq \frac{\epsilon}{n+1}\right) \leq \frac{1}{m\epsilon^2}$, where $Y = \frac{\sum_{i=1}^{m} X_i}{m}$.

(Q8) Let $S \subseteq 2^{[n]}$ where $[n] = \{1, \ldots, n\}$ satisfying the following properties: (i) $|S| = m$, and (ii) $\forall \sigma \in S$ we have $|\sigma| = k$. Show that for all $0 < p < 1$, there exist a set $A_p \subseteq [n]$ such that, for all $\sigma \in S$, $A_p \cap \sigma \neq \emptyset$ and
$$|A_p| \leq np + m(1-p)^k.$$
If $n \leq mk$, then using the above result show that there exist a set $A \subseteq [n]$ such that, for all $\sigma \in S$, $A \cap \sigma \neq \emptyset$ and $|A| = O\left(\frac{n}{k} \log\left(\frac{mk}{n}\right)\right)$.

(Q9) Let X and Y are two independent random variables having moment generating functions $M_X(t) = \left(\frac{e^t+2}{3}\right)^6$ and $M_Y(t) = e^{3e^t - 3}$. Compute $\mathbb{P}(X + Y \geq 3)$ and $\mathbb{P}(XY = 0)$.

(Q10) Suppose you are given with two distinct coins whose probability of heads are $1/2$ and $2/3$. How can you detect the coin with probability $1/2$ with probability $1 - \delta$ for any given $\delta > 0$?