INDIAN STATISTICAL INSTITUTE

End Semestral Examination

M. Tech (CS) - I Year, 2018-2019 (Semester - I)

Probability and Stochastic Processes

Problem Sheet I

(Q1) Two friends decide to meet at a venue at a given time. Each comes to the venue with a delay between 0 and 1 hour, with all pairs of delays being equally likely. The first to arrive will wait for 20 minutes and will leave if the other has not yet arrived. What is the probability that they will meet?

(Q2) $\Pr(A \cup B)$ gives the probability that at least one of the events A and B will occur. Express mathematically what do you mean by the probability of the event that exactly one of the events A and B will occur.

(Q3) Suppose a candidate is participating in a game show. The candidate is given the choice of three doors (A, B and C) – behind one door is a car; behind the other two, goats. The candidate picks a door at random, say A, but the chosen door is not opened immediately. The host of the game show, who knows what is behind the doors, opens another door, say C, which shows a goat. The host then says to the candidate, ”Do you want to pick door B?” Is it to the advantage of the candidate to switch his/her choice? Give proper arguments in favour of your answer.

(Q4) Choose a number uniformly at random in the range $[1, 100,000]$. Determine the probability that the number chosen is divisible by one or more of 4, 6, and 9.

(Q5) Suppose that we roll ten standard six-sided dice. What is the probability that their sum will be divisible by 6, assuming that the rolls are independent?

(Q6) If n married couples are seated at random at a round table, compute the probability that no wife sits next to her husband.

(Q7) Consider a sequence of n heads and m tails. Assume that all the orderings of the n heads and m tails are equally likely. A *run* of heads (or tails) is a sequence of consecutive heads (or tails). Determine the probability that there will be exactly r runs of wins.

(Q8) A single bit, either a 0 or a 1, is being transmitted through a series of n relays before it arrives to its destination from the source. Each relay flips the bit independently with probability p. Find the probability that the correct bit is received at the destination.

(Q9) x per cent of the surface of a sphere is coloured black and the rest is white. Find an upper bound on x such that irrespective of the manner in which the colours are distributed, it is possible to inscribe a cube in the sphere with all its vertices white.

(Q10) Independent trials, consisting of rolling a pair of fair dice, are performed. What is the probability that an outcome of 5 appears before an outcome of 7 when the outcome of a roll is the sum of dice?

(Q11) A permutation is a one-to-one and onto function $\pi : A \to A$, where $A = \{1, 2, \ldots, n\}$. A fixed-point of a permutation is defined as an index i for which $\pi(i) = i$. A cycle of a permutation is defined as a set of indices $\{i_1, i_2, \ldots, i_k\} \in A$, $k \leq n$, such that $\pi(i_1) = i_2$, $\pi(i_2) = i_3$, ..., $\pi(i_{k-1}) = i_k$, and $\pi(i_k) = i_1$.

(i) Design a method that generates any permutation of A with equal probability $\frac{1}{n!}$. Prove your result.

(ii) What is the expected number of fixed points in a random permutation?

(iii) What is the expected number of cycles in a random permutation?

(Q12) If E and F are independent, then so are (i) E and $\overline{F}$; and (ii) $\overline{E}$ and $\overline{F}$.

(Q13) If $\Pr(A) > 0$, show that $\Pr(AB \mid A) \geq \Pr(AB \mid A \cup B)$.

(Q14) Prove that if E_1, E_2, ..., E_n are independent events, then

$$\Pr\left(E_1 \cup E_2 \cup \cdots \cup E_n\right) = 1 - \prod_{i=1}^{n} \left(1 - \Pr(E_i)\right)$$

(Q15) Independent trials, each resulting in a success with probability p or a failure with probability $q = 1 - p$, are performed. Compute the probability that a run of n consecutive successes occurs before a run of m consecutive failures.

(Q16) A parallel system having n components functions when at least one of the components functions. A component i, independent of other components, functions with probability p_i, $i = 1, \ldots, n$. What is the probability that the system functions?

(Q17) Let X be a random variable defined over a sample space Ω such that $\mathrm{E}[X] = \mu$. Show that $\Pr(X \geq \mu) > 0$ and $\Pr(X \leq \mu) > 0$. [Hints: Can you try to prove using contradiction?]

(Q18) Let A, B, and C be independent events, with $\Pr(C) > 0$. Prove that A and B are conditionally independent given C.