INDIAN STATISTICAL INSTITUTE

Class Test II

M Tech (CS) – I Year, 2017-2018 (Semester - I)

Probability and Stochastic Processes

Date: 27.09.2018 Maximum Marks : 30 Duration : 1.0 hours

Note: The question paper is of 40 marks. Answer as much as you can, but the maximum you can score is 30. Answer a question within its allotted box.

Course: (M Tech/JRF/PLP) ______________________

Name: ______________________ **Roll Number:** ______________

(Q1) Let X and Y be independent variables, X being equally likely to take any value in $\{0, 1, \ldots, m\}$, and Y similarly in $\{0, 1, \ldots, n\}$. Find the mass function of $Z = X + Y$. [10]

Ans. Let $[k] = \{0, \ldots, k\}$. Note that $X \in [m], Y \in [n]$ and $Z \in [m+n]$. For $z \in [m+n]$, let N_z be the number of solutions to $x + y = z$ such that $x \in [m]$ and $y \in [n]$.

$$\begin{aligned}
\Pr(Z = z) &= \sum_{x\in[m],y\in[n]:x+y=z} \Pr(X = x, Y = y) \\
&= \sum_{x\in[m],y\in[n]:x+y=z} \Pr(X = x) \cdot \Pr(Y = y) \\
&= \frac{N_z}{(m+1)(n+1)}.
\end{aligned}$$

(Q2) Define $\text{var}(Y \mid X)$, the conditional variance of Y given X. Show that $\text{var}(Y) = \text{E}(\text{var}(Y \mid X)) + \text{var}(\text{E}(Y \mid X))$. [10]

Ans. $\text{var}(Y \mid X) = \text{E}\left((Y - \text{E}(Y \mid X))^2 \mid X\right) = \text{E}\left(\text{E}(Y^2 \mid X) - (\text{E}(Y \mid X))^2\right)$. For the second part,

$$\begin{aligned}
& \text{E}(\text{var}(Y \mid X)) + \text{var}(\text{E}(Y \mid X)) \\
= & \left[\text{E}\left(\text{E}(Y^2 \mid X) - (\text{E}(Y \mid X))^2\right)\right] + \left[\text{E}\left((\text{E}(Y \mid X))^2\right) - (\text{E}(\text{E}(Y \mid X)))^2\right] \ (\because \text{var}(Y) = \text{E}(Y^2) - (\text{E}(Y))^2) \\
= & \text{E}\left(\text{E}(Y^2 \mid X)\right) - \text{E}\left((\text{E}(Y \mid X))^2\right) + \text{E}\left((\text{E}(Y \mid X))^2\right) - (\text{E}(\text{E}(Y \mid X)))^2 \\
= & \text{E}\left(\text{E}(Y^2 \mid X)\right) - (\text{E}(\text{E}(Y \mid X)))^2 \\
= & \text{E}(Y^2) - (\text{E}(Y))^2 \ (\because \text{E}(\text{E}(Y \mid X)) = \text{E}(Y)) \\
= & \text{var}(Y)
\end{aligned}$$

(Q3) A coin shows heads with probability p. Let X_n be the number of flips required to obtain a run of n consecutive heads. Show that $\text{E}(X_n) = \sum_{k=1}^{n} p^{-k}$. [10]

Ans. Let X_k be the random variable that denotes the number of tosses to get k consecutive heads. Note that $\text{E}[X_1] = \frac{1}{p}$. Observe that, for $n \geq 2$,

$$\text{E}(X_n \mid X_{n-1}) = p(X_{n-1} + 1) + (1-p)(X_{n-1} + 1 + \text{E}(X_n))$$

$$
\begin{aligned}
\mathrm{E}\left(\mathrm{E}\left(X_n \mid X_{n-1}\right)\right) &= \mathrm{E}\left(p(X_{n-1}+1) + (1-p)(X_{n-1}+1+\mathrm{E}(X_n))\right) \\
\mathrm{E}(X_n) &= p(\mathrm{E}(X_{n-1})+1) + (1-p)(\mathrm{E}(X_{n-1})+1+\mathrm{E}(X_n)) \\
\mathrm{E}(X_n) &= \frac{1}{p}(\mathrm{E}(X_{n-1})+1)
\end{aligned}
$$

One can get the desired result after solving the above recurrence.

(Q4) Let $X_1, \ldots, X_n$ be independent random variables with $\Pr\left(X_i = 1\right) = \Pr\left(X_i = -1\right) = \frac{1}{2}$. Let $X = \sum_{i=1}^{n} X_i$. Show that for all $a > 0$, $\Pr\left(X \geq a\right) \leq e^{-a^2/2n}$. [10]

Ans. Please refer Mitzenmacher and Utfal Theorem 4.7.