INDIAN STATISTICAL INSTITUTE

Class Test I

M Tech (CS) – I Year, 2017-2018 (Semester - I)

Probability and Stochastic Processes

Date: 27.08.2018 | Maximum Marks : 30 | Duration : 1.0 hours

Note: The question paper is of 40 marks. Answer as much as you can, but the maximum you can score is 30. Answer a question within its allotted box.

Course: (M Tech/JRF/PLP) ______________________________

Name: ______________________ **Roll Number:** ______________

(Q1) You are given that at least one of the events A_r, $1 \leq r \leq n$, is certain to occur, but certainly no more than two occur. If $\Pr(A_r) = p$, and $\Pr(A_r \cap A_s) = q$, $r \neq s$, show that $p \geq \frac{1}{n}$ and $q \leq \frac{2}{n}$. [10]

At least one of the events A_r, $1 \leq r \leq n$, is certain to occur. So, $1 = \Pr\left(\bigcup_{i=1}^{n} A_r\right)$. Using Boole's inequality or the union bound, we have

$$1 = \Pr\left(\bigcup_{i=1}^{n} A_r\right) \leq \sum_{i=1}^{n} \Pr(A_r) = np$$

$$\text{So,} \quad p \geq \frac{1}{n}$$

$$\begin{aligned}
\Pr\left(\bigcup_{i=1}^{n} A_r\right) &= 1 \\
\text{or, } \sum_{r} \Pr(A_r) - \sum_{r<s} \Pr(A_r \cap A_s) &= 1 \text{ (as no more than two events occur)} \\
\text{or, } np - \frac{n(n-1)q}{2} &= 1 \qquad (1)
\end{aligned}$$

Because of $p \geq \frac{1}{n}$ and Equation (1), we have

$$\begin{aligned}
\frac{n(n-1)q}{2} &= np - 1 \\
\text{or, } \frac{n(n-1)q}{2} &\leq n - 1 \text{ (as } p \leq 1) \\
\text{or, } q &\leq \frac{2}{n}
\end{aligned}$$

(Q2) Assume that the events A_1, A_2, A_3 and A_4 are independent. Also, $\Pr(A_3 \cap A_4) > 0$. Show that

$$\Pr(A_1 \cup A_2 \mid A_3 \cap A_4) = \Pr(A_1 \cup A_2)$$

. [10]

First note that,

$$\Pr(A_1 \cup A_2 \mid A_3 \cap A_4) = \Pr(A_1 \mid A_3 \cap A_4) + \Pr(A_2 \mid A_3 \cap A_4) - \Pr(A_1 \cap A_2 \mid A_3 \cap A_4)$$

We have to now compute $\Pr(A_1 \mid A_3 \cap A_4)$, $\Pr(A_2 \mid A_3 \cap A_4)$ and $\Pr(A_1 \cap A_2 \mid A_3 \cap A_4)$.

$$\Pr(A_1 \mid A_3 \cap A_4) = \frac{\Pr(A_1 \cap A_3 \cap A_4)}{\Pr(A_3 \cap A_4)} = \frac{\Pr(A_1)\Pr(A_3)\Pr(A_4)}{\Pr(A_3)\Pr(A_4)} = \Pr(A_1)$$

We do likewise to obtain $\Pr(A_2 \mid A_3 \cap A_4) = \Pr(A_2)$ and $\Pr(A_1 \cap A_2 \mid A_3 \cap A_4) = \Pr(A_1 \cap A_2)$ and then plug in the values as follows.

$$\begin{aligned}
\Pr(A_1 \cup A_2 \mid A_3 \cap A_4) &= \Pr(A_1 \mid A_3 \cap A_4) + \Pr(A_2 \mid A_3 \cap A_4) - \Pr(A_1 \cap A_2 \mid A_3 \cap A_4) \\
&= \Pr(A_1) + \Pr(A_2) - \Pr(A_1 \cap A_2) \\
&= \Pr(A_1 \cup A_2)
\end{aligned}$$

(Q3) There are n letters marked for n envelopes. The letters are mixed up and put randomly inside the envelopes. A *match* occurs if a letter goes into the envelope it is marked for. What is the probability of exactly k matches? [10]

Let $P(i)$ denote the probability of *derangement* of i letters (which we know from our class). For the probability of exactly k matches, consider a fixed set of k letters. The probability $\mathcal{P}$ that this set of k letters (and only this set) match, and by implication the other $n-k$ letters *derange* is

$$\mathcal{P} = \frac{1}{n}\frac{1}{n-1}\cdots\frac{1}{n-(k-1)}P(n-k) = \frac{(n-k)!}{n!}P(n-k).$$

Now as there are $\binom{n}{k}$ choices of the set of k letters, the requisite probability is

$$\binom{n}{k}\mathcal{P} = \frac{n!}{k!(n-k)!}\frac{(n-k)!}{n!}P(n-k) = \frac{P(n-k)}{k!}$$

(Q4) There are n urns of which the i-th urn contains $i-1$ red balls and $n-i$ blue balls. You pick an urn at random and remove two balls at random without replacement. Find the probability that (a) the second ball is blue; (b) the second ball is blue given the first ball is blue. [10]

Let C_i denote the color of the i-th ball picked. Each urn contains $n-1$ balls, so there are $n(n-1)$ balls in all, of which $\frac{n(n-1)}{2}$ are blue (B) and $\frac{n(n-1)}{2}$ are red (R).

(a) The answer is $\frac{1}{2}$. You can see it using total probability.

(b) We seek $\Pr(C_2 = B \mid C_1 = B)$ by conditioning on the choice of the urn. $\Pr(C_1 = B) = \frac{\frac{n(n-1)}{2}}{n(n-1)} = \frac{1}{2}$.

$$\Pr(C_2 = B \mid C_1 = B) = \frac{\Pr(C_1, C_2 = B)}{\Pr(C_1 = B)} = \frac{\Pr(C_1, C_2 = B)}{\frac{1}{2}}. \tag{2}$$

Now, to find $\Pr(C_1, C_2 = B)$, we condition on the choice of the urn as

$$\begin{aligned} \Pr(C_1, C_2 = B) &= \sum_{i=1}^{n} \Pr(C_1, C_2 = B \mid i\text{-th urn was chosen}) \cdot \Pr(i\text{-th urn was chosen}) \\ &= \frac{1}{n} \sum_{i=1}^{n} \frac{n-i}{n-1} \cdot \frac{n-i-1}{n-2} \quad [\text{as Pr(i-th urn was chosen)} = \frac{1}{n}] \end{aligned}$$

Replace the above in Equation (2), to get

$$\Pr(C_2 = B \mid C_1 = B) = \frac{\Pr(C_1, C_2 = B)}{\Pr(C_1 = B)} = \frac{\left(\sum_{i=1}^{n} \frac{(n-i)(n-i-1)}{n(n-1)(n-2)}\right)}{\frac{1}{2}} = \frac{\frac{2n(n-1)(n-2)}{6n(n-1)(n-2)}}{\frac{1}{2}} = \frac{2}{3}$$