INDIAN STATISTICAL INSTITUTE

Mid Semestral Examination

M. Tech (CS) - I Year, 2010-2011 (Semester - I)

Discrete Mathematics

Date : 23.09.2010 Maximum Marks : 60 Duration : 3.0 Hours

Note: This is a **cheat sheet based** examination. You can carry with yourself **two A4 sized sheets** with your name and roll number written neatly on top of both the sheets. You have to submit the cheat sheets after the examination is over. Cheat sheets cannot be shared.

Answer as much as you can, but the maximum you can score is 60.

This is a 2 page question paper with 8 questions.

(Q1) Prove or disprove the following statement. Given two non-negative functions $f(n)$ and $g(n)$, either $f(n) = O(g(n))$ or $g(n) = O(f(n))$. [6]

(Ans:) The statement in the question is not true. The big-oh is not like $<$ such that two functions can always be in a big-oh relation. Consider the following example with $f(n)$ and $g(n)$.

$$f(n) = \begin{cases} n & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

$$g(n) = \begin{cases} n & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

(Q2) Let P be a set of n points lying on the circumference of a circle such that if lines are drawn connecting every point to every other point, then no three of these lines intersect in a single point inside the circle. Let A be the set of all points of intersection of the lines in the interior of the circle. Note that, the points of P are not included in A. Derive an expression for the cardinality of the set A. [12]

[Hint: Can you define functions and count?]

(Ans:) A is the set of all points of intersection of the lines in the interior of the circle. Note that the points of P do not belong to A. Define

$$B = \{\{p_1, p_2, p_3, p_4\} \subseteq P \mid p_1, p_2, p_3, p_4 \text{ are all distinct.}\}$$

It is easy to see that $|B| = \binom{n}{4}$. We can define a function $f : A \to B$ as follows. Let $a \in A$ be an intersection point. Then, a lies on exactly two lines (and not more as per problem statement). Consider the set $b \in B$ containing the four end points of these two line segments. Define $f(a) = b$. Now, we can show that f is both one-to-one and onto. Thus, f is a bijection and so $|A| = |B| = \binom{n}{4}$.

(Q3) A binary relation R on a set X, denoted as (X, R) is a subset of the Cartesian product $X \times X$. We use $\sim aRb$ to denote that (a, b) is not in R.

A binary relation is irreflexive if $\sim aRa, \forall a \in X$.

A binary relation is asymmetric if $aRb \Rightarrow \sim bRa, \forall a, b \in X$.

A binary relation is antisymmetric if aRb and bRa implies $a = b, \forall a, b \in X$.

Now, prove the following statement. *A binary relation is irreflexive, transitive and antisymmetric if and only if it is transitive and asymmetric.* [8]

(Ans:) First consider the proof of the statement: *a binary relation, R is irreflexive, transitive and antisymmetric $\Rightarrow$ R is transitive and asymmetric.* We just need to show that R is asymmetric. Suupose, that (X, R) is irreflexive, transitive, and antisymmetric. Consider $a, b \in X$ such that aRb and bRa. As R is antisymmetric, then $a = b$. But as R is irreflexive, so $\sim aRa$. So, aRb and bRa cannot both hold together and $aRb \Rightarrow \sim bRa, \forall a, b \in X$ making R asymmetric.

Now cosider proving *R is transitive and asymmetric $\Rightarrow$ R is irreflexive, transitive and antisymmetric.* We need to show that R is irreflexive and antisymmetric. As R is transitive and asymmetric, aRb and bRa cannot both hold and obviously R is antisymmetric. It is easy to see that asymmetry implies irreflexivity.

(Q4) (i) Let us consider our disrete mathematics class. If Shaondip is late, then Kalyan is late, and if both Shaondip and Kalyan are late, then the class is boring. Suppose that the class is not boring! What can you conclude about Shaondip?

[Hint: Formulate the problem in terms of logic.]

(Ans:) Let p = *Shaondip is late*, q = *Kalyan is late* and r = *The class is boring*. Now, consider the compound statement S = *If Shaondip is late, then Kalyan is late, and if both Shaondip and Kalyan are late, then the class is boring.* So, $S = (p \Rightarrow q) \wedge [(p \wedge q) \Rightarrow r]$. The truth table for S is as follows:

p	q	r	S
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

It is given that the class is not boring. Thus, S and r takes TRUE and FALSE values respectively. So, row numbers 1, 2, 3, 4, 5 and 7 can be eliminated. What remains are the entries for the 6th and 8th rows. In both of these rows, p is FALSE. So, our conclusion is that Shaondip is never late.

(ii) Write the following statement in predicate logic and then negate it. Clearly mention what is your domain and predicates.

Let x and y be real numbers. If x is rational and y is irrational, then $x + y$ is irrational.

(Ans:) Let $P(x) : x$ is rational in the domain of real number. The statement given in the problem translates into predicate logic as follows:

$$\forall x \forall y ((P(x) \wedge \sim P(y)) \Rightarrow \sim P(x + y))$$

For a simpler form, let $A = (P(x) \wedge \sim P(y)$, and let $B = P(x+y)$. So, the statement translates as $\forall x \forall y (A \Rightarrow \sim B)$. The negation of the above statement gives us $\exists x \exists y \sim (A \Rightarrow \sim B)$. We can verify that $\sim (A \Rightarrow \sim B)$ and $A \wedge B$ are logically equivalent. So, $\exists x \exists y \sim (A \Rightarrow \sim B)$ becomes

$$\exists x \exists y ((P(x) \wedge \sim P(y)) \wedge (P(x+y))).$$

[4+(3+3)=10]

(Q5) (i) Find out the number of ways in which you can distribute n distinguishable balls into k distinguishable cells where no cell remains empty. Give justifications for your answer.

(Ans:) First, recall that the number of ways to distribute n distinguishable balls into k indistinguishable cells where no cell remains empty is the Stirling number of the second kind and is given by

$$S(n,k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n.$$

Now, to make the k cells distinguishable, we just need to permute the k cells in $k!$ ways. So, the answer is $k!S(n,k)$.

(ii) Find out the number of ways in which you can distribute n distinguishable balls into k indistinguishable cells where cells can be empty. Give justifications for your answer.

(Ans:) Again, consider $S(n,k)$. Now, cells can be empty, i.e. one cell can be non-empty, OR two cells can be non-empty, OR three cells can be non-empty, ... Thus, the answer is $\sum_{i=1}^{k} S(n,i)$.

[4+6=10]

(Q6) A box contains 10 blue balls, 20 red balls, 8 green balls, 15 yellow balls and 25 white balls. How many balls must we choose to ensure that we have 12 balls of the same colour? [5]

(Ans:) This is an easy application of pigeonhole principle. There are five categories of balls. Consider them to be pigeon holes. Each one should have 11 balls so that the next ball drawn should ensure 12 balls of the same colour. But, blue and green balls are less in number. So, drawing the entire lot would suffice. Thus, the number of balls to be chosen so that we have 12 balls of the same colour is $10 + 11 + 8 + 11 + 11 + 1 = 52$.

(Q7) Prove that if $n^2 + 1$ points are placed in an equilateral triangle (the region inside as well as the perimeter) of side length 1, then there are two points whose distance is at most $\frac{1}{n}$. [8]

[Hint: Start with small values of n and then try to generalize.]

(Ans:) Starting with $n = 2$, we can break the equilateral triangle into 4 equilateral triangles as shown in Figure 7. Now, with 5 points, there exists a triangle which should have at least 2 points. How far can they be apart? They can be located at the two vertices at a distance of $\frac{1}{2}$. For $n = 3$, we can break the original equilateral triangle into $1+3+5 = 9$ equilateral triangles (as shown in Figure 7) and apply the pigeonhole principle with 9 holes(triangle) and 10 pigeons so that there exists a triangle with 2 points which can be maximum apart by a distance of at most $\frac{1}{3}$. We can generalize this to $1 + 3 + 5 + \ldots + (2n-1) = n^2$ equilateral triangles, so that if $n^2 + 1$ points are placed, then there exists two points that are at most $\frac{1}{n}$ distance apart.

(Q8) Consider a building having a staircase with n stairs. In how many ways can a person climb the staircase, if she can climb by 1 or by 2 stairs in each step? Find out a closed form expression in terms of n. [8]

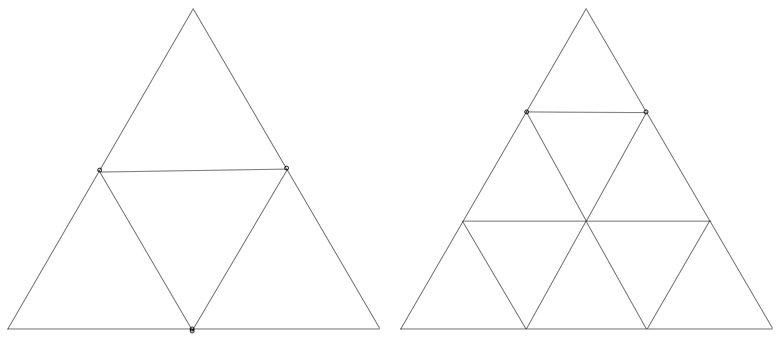

Figure 1: Triangle partition.

(Ans:) Let $F(n)$ be the number of ways in which a person can climb the staircase of n stairs, if she can climb by 1 or by 2 stairs in each step. She can reach the n-th stair from the $(n-1)$-th stair or the $(n-2)$-th stair. The number of ways in which $(n-1)$ stairs and $(n-2)$ stairs can be climbed are $F(n-1)$ and $F(n-2)$ respectively. For the boundary conditions, we have $F(1) = 1$ and $F(2) = 2$ as two stairs can be climbed in two ways, a single stair followed by another single stair and two stairs at a time. So, we have the following recurrence:

$$F(n) = \begin{cases} 1 & \text{if } n = 1; \\ 2 & \text{if } n = 2; \\ F(n-1) + F(n-2) & \text{if } n > 2. \end{cases}$$

The recurrence is almost a Fibonacci recurrence with a small change in the boundary conditions which would affect the constants.