INDIAN STATISTICAL INSTITUTE

Class Test II

M. Tech (CS) - I Year, 2010-2011 (Semester - I)

Discrete Mathematics

Date : 14.12.2010 Maximum Marks : 30 Duration : 1.5 Hours

Note: You may answer any part of any question, but maximum you can score is 30.

(Q1) Let $G = (V, E)$ be an undirected graph with k components. Prove or disprove the following statement: $|E| \geq |V| - k$. [5]

(Ans:) Consider any connected component i and let there be v_i vertices and e_i edges. As this component is connected, we have $e_i \geq v_i - 1$. We sum it over all connected components, i.e. $\sum_{i=1}^{k} e_i \geq \sum_{i=1}^{k}(v_i - 1) = |E| \geq |V| - k$.

(Q2) Let T be a spanning tree of K_n, the complete graph on n vertices, and let $p = P(T)$ be its Prüfer code. Let x_i be the number of times the vertex i appears in the sequence p, $i = 1, \ldots, n$. Prove or disprove the following statement: "$\deg_T(i) = x_i + 1$ holds for all i". [5]

(Ans:) The statement is true. The basic idea of Prüfer code is to tear off the leaves of T one by one until the tree is reduced to a single edge. While doing so, we record the neighbor of the leaf just torn off to generate the Prüfer code. So, we have two types of vertices: (Type 1:) the $n - 2$ vertices that have been torn off; (Type 2:) the remaining two vertices of the single edge. Some of the vertices of Type 1 are leaf vertices of T and hence these vertices do not occur in the Prüfer code, i.e. $x_i = 0$ and the formula trivially holds for them. For any other vertex i of Type 1, if it occurs x_i times, then it surely has x_i neighbors because i was recorded in the Prüfer code each time a neighbor was torn off. As i is a vertex of Type 1, it has a neighbor that was recorded when i was torn off. Thus, $\deg_T(i) = x_i + 1$ holds. A vertex i of Type 2 surely has x_i neighbors because i was recorded in the Prüfer code each time a neighbor was torn off. And $\deg_T(i) = x_i + 1$ is true because one edge is still incident to vertex i which of Type 2.

(Q3) A jailer has a problem on his hands. He has to put some criminals in cells of the jail. The criminals are notorious and can beat one another to death. If any criminal dies inside the cell, then the jailer will lose his job. In this scenario, the jailor thought of putting each criminal in a cell. But, his boss wants it to be done using the minimum number of cells. The only saving grace for the jailer is that the criminals fight according to the following pattern: (i) a criminal does not beat himself; (ii) if a criminal A does not beat a criminal B, and criminal B does not beat criminal C, then criminal A does not beat criminal C (criminals A, B and C are distinct). Help the jailer save his job by solving this problem efficiently. [10]

(Ans:) Let a criminal be a vertex of a directed graph G. The edges of G are formed using a *friendship* relation. If a criminal A does not beat a criminal B, i.e., A knows that B is a friend, then we add a directed edge from A to B. Note that, B might not have A to be a friend. Now, look into the pattern (ii). How does it reflect in G? If a criminal A knows B to be a friend and B knows C to be a friend, then A knows C to be a friend. This implies a directed path in G from A to

C. So, which all criminals can be put in a cell? Criminals who know each other to be friends can be put in a cell. All pairs of vertices (criminals) that are mutually reachable are friends. What are these vertices in G - nothing but the *strongly connected component* in G. So, the jailer finds out the *strongly connected component* in G and reports the answer to the boss!

(Q4) Let $G = (V, E)$ with $|V| = n$ be a connected graph. Let the maximum independent set of G be $\beta(G)$ and the chromatic number of G be $\chi(G)$. Prove that $n \leq \beta(G)\chi(G)$. Use this result to show that $\beta(G) \geq n/4$ for a planar graph. [8+2=10]

[Hints: A problem in graph theory may have its solution in combinatorics!]

(Ans:) Partition the vertices of G by their colors. So, n vertices are distributed among $\chi(G)$ colors. Now, apply pigeonhole principle with n vertices as pigeons and $\chi(G)$ colors as holes. So, one of the colors must contain $n/\chi(G)$ vertices. By the definition of coloring in a graph, these $n/\chi(G)$ vertices should be pairwise nonadjacent, i.e. they should not have edges. So, these $n/\chi(G)$ vertices form an independent set. Therefore, the maximum independent set $\beta(G) \geq n/\chi(G)$. So, $n \leq \beta(G)\chi(G)$.

The famous four color theorem for a planar graph states that a planar graph can be colored using at most four colors, i.e. $\chi(G) \leq 4$. Using the relation $\beta(G) \geq n/\chi(G)$ and $\chi(G) \leq 4$, we have $\beta(G) \geq n/4$ for a planar graph.

(Q5) Prove that the pair, ($\Rightarrow$, $\vee$) is not alone adequate to express all truth functions. [5]

(Ans:) With ($\Rightarrow$, $\vee$), it is not possible to implement $\sim$.

(Q6) Let L be a formal axiomatic theory for the propositional calculus. Prove that for any well formed formulas A and B, the followings are theorems of L.
(i) $A \Rightarrow (\sim B \Rightarrow \sim (A \Rightarrow B))$; (ii) $(\sim A \Rightarrow (A \Rightarrow B))$

[5+5=10]

[Hints: Use Deduction Theorem.]

(Ans:) (i) We need to show $\vdash_L A \Rightarrow (\sim B \Rightarrow \sim (A \Rightarrow B))$.

(1)	A	(hypothesis)
(2)	$A \Rightarrow B$	(hypothesis)
(3)	B	(1, 2, MP)
(4)	$\vdash_L A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$	(Use Deduction theorem twice)
(5)	$((A \Rightarrow B) \Rightarrow B) \Rightarrow (\sim B \Rightarrow \sim (A \Rightarrow B))$	(by known theorem of contrapositivity)
(6)	$A \Rightarrow (\sim B \Rightarrow \sim (A \Rightarrow B))$	(using $\{A \Rightarrow B, B \Rightarrow C\} \vdash_L A \Rightarrow C$)

(ii) We need to show $\vdash_L (\sim A \Rightarrow (A \Rightarrow B))$.

(1)	$\sim A$	hypothesis
(2)	A	hypothesis
(3)	$A \Rightarrow (\sim B \Rightarrow A)$	Axiom 1
(4)	$\sim A \Rightarrow (\sim B \Rightarrow \sim A)$	Axiom 1
(5)	$\sim B \Rightarrow A$	2, 3 and MP
(6)	$\sim B \Rightarrow \sim A$	1, 4 and MP

(7)	$(\sim B \Rightarrow \sim A) \Rightarrow ((\sim B \Rightarrow A) \Rightarrow B)$	Axiom 3
(8)	$(\sim B \Rightarrow A) \Rightarrow B$	6, 7 and MP
(9)	B	5, 8 and MP

Thus, $\{\sim A,\, A\} \vdash_L B$. Apply deduction theorem twice to get $(\sim A \Rightarrow (A \Rightarrow B))$.