INDIAN STATISTICAL INSTITUTE

Class Test I

M. Tech (CS) - I Year, 2010-2011 (Semester - I)

Discrete Mathematics

Date : 21.09.2010　　Maximum Marks : 30　　Duration : 1.5 Hours

Note: You may answer any part of any question, but maximum you can score is 30.

(Q1) Prove or disprove the following statement. Given non-negative functions $f_1(n)$, $f_2(n)$, $g_1(n)$, $g_2(n)$ such that $f_1(n) = O(g_1(n))$, $f_2(n) = O(g_2(n))$, and for all integers $n \geq 0$, $g_1(n) < g_2(n)$, then $f_1(n) < f_2(n)$. [5]

(Ans:) The statement is false. Consider the following example: $f_1(n) = n^2$, $f_2(n) = n$, $g_1(n) = n^3$ and $g_2(n) = n^3 + 1$.

(Q2) Let R be a partial order on X and $Y \subseteq X$. If every two elements in Y are comparable, then Y is a *chain* of X. If no two distinct elements of Y are comparable, then Y is an *antichain* of X. Now, prove or disprove the following statement. In any partial order on a set X of $mn+1$ elements, there exists a chain of size $m+1$ or an antichain of size $n+1$. [8]

[Hints: Can you find an application of pigeonhole principle?]

(Ans:) Suppose, there is no chain of size $m+1$ in X. We would show that this implies an antichain of size $n+1$. Define a function $f : X \to \{1, 2, \ldots, m\}$ with $f(x)$ equal to the greatest number of elements in a chain with greatest element x. By the pigeonhole principle, there exists a subset $Y' \subseteq X$ of $\lceil \frac{mn+1}{m} \rceil = n+1$ elements that maps to the same element under the function f. The trick was in the definition of f. By the definition of f, the elements in Y' are incomparable. So, Y' forms an antichain of size $n+1$.

(Q3) A codeword consists of at least one of each of the digits 0, 1, 2, 3 and 4, and has length 6. How many such codewords are there? [5]

[Formulation of the problem as a generating function and mentioning relevant exponent would be enough.]

(Ans:) Each digit has at least one occurrence. There are five digits in all and the length of the codeword is 6. So, each digit can occur at most twice. Therefore, the contribution of each term to the generating function will be $\left(x + \frac{x^2}{2!}\right)$. Thus, the generating function will be $\left(x + \frac{x^2}{2!}\right)^5$. As order matters in a code-word, this is an application of exponential generating function. We have to collect the co-efficient of $\frac{x^6}{6!}$ in this expression. The answer turns out to be 1800.

(Q4) Prove or disprove the following statement: $(A \Leftrightarrow B) \Leftrightarrow C$ and $A \Leftrightarrow (B \Leftrightarrow C)$ are logically equivalent. [5]

(Ans:) To show that $(A \Leftrightarrow B) \Leftrightarrow C$ and $A \Leftrightarrow (B \Leftrightarrow C)$ are logically equivalent, we have to show that $((A \Leftrightarrow B) \Leftrightarrow C) \Leftrightarrow (A \Leftrightarrow (B \Leftrightarrow C))$ is a tautology.

A	B	C	$A \Leftrightarrow B$	$B \Leftrightarrow C$	$(A \Leftrightarrow B) \Leftrightarrow C$	$A \Leftrightarrow (B \Leftrightarrow C)$	$((A \Leftrightarrow B) \Leftrightarrow C) \Leftrightarrow (A \Leftrightarrow (B \Leftrightarrow C))$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	F	F	F	T
T	F	F	F	T	T	T	T
F	T	T	F	T	F	F	T
F	T	F	F	F	T	T	T
F	F	T	T	F	T	T	T
F	F	F	T	T	F	F	T

(Q5) Find out the number of integers between 1 and $10,000$ that are divisible by any of the integers 2, 3, 5 and 7. [5]

(Ans:) Let S_2, S_3, S_5 and S_7 denote the set of integers that are divisible by 2, 3, 5 and 7 respectively. Therefore, $|S_2| = \lfloor\frac{10000}{2}\rfloor = 5000$, $|S_3| = \lfloor\frac{10000}{3}\rfloor = 3333$, $|S_5| = \lfloor\frac{10000}{5}\rfloor = 2000$ and $|S_7| = \lfloor\frac{10000}{7}\rfloor = 1428$. Next, we need to look at numbers between 1 and $10,000$ that are divisible by products of the said numbers. The numbers that are divisible by both 2 and 3 are divisible by their product and hence is nothing but the set $S_2 \cap S_3$. So, we have $|S_2 \cap S_3| = \lfloor\frac{10000}{2\times3}\rfloor = 1666$, $|S_2 \cap S_5| = \lfloor\frac{10000}{2\times5}\rfloor = 1000$, $|S_2 \cap S_7| = \lfloor\frac{10000}{2\times7}\rfloor = 714$, $|S_3 \cap S_5| = \lfloor\frac{10000}{3\times5}\rfloor = 666$, $|S_3 \cap S_7| = \lfloor\frac{10000}{3\times7}\rfloor = 476$, $|S_5 \cap S_7| = \lfloor\frac{10000}{5\times7}\rfloor = 285$. Similarly, we need to look at products of three numbers. $|S_2 \cap S_3 \cap S_5| = \lfloor\frac{10000}{2\times3\times5}\rfloor = 333$, $|S_2 \cap S_3 \cap S_7| = \lfloor\frac{10000}{2\times3\times7}\rfloor = 238$, $|S_2 \cap S_5 \cap S_7| = \lfloor\frac{10000}{2\times5\times7}\rfloor = 142$, $|S_3 \cap S_5 \cap S_7| = \lfloor\frac{10000}{3\times5\times7}\rfloor = 95$. Also, we need to look at product of four numbers. $|S_2 \cap S_3 \cap S_5 \cap S_7| = \lfloor\frac{10000}{2\times3\times5\times7}\rfloor = 47$.

The number of integers between 1 and $10,000$ that are divisible by any of the integers 2, 3, 5 and 7 is $S_1 \cup S_2 \cup S_3 \cup S_4 = (5000 + 3333 + 2000 + 1428) - (1666 + 1000 + 714 + 666 + 476 + 285) + (333 + 238 + 142 + 95) - 47 = 7715$.

(Q6) For $n = 2^{2^k}$, $k \geq 1$, do proper substitution to bring the following recurrence to a form you know and has been discussed in the class. You get extra credit for solving it.

$$T(n) = \begin{cases} d & \text{if } n = 2; \\ 2T(\sqrt{n}) + b\log n & \text{if } n > 2. \end{cases}$$

[5+2]

(Ans:) With $n = 2^{2^k}$, $2^k = \log n$ and the recurrence after substitution is as follows:

$$T(2^{2^k}) = \begin{cases} d & \text{if } k = 0; \\ 2T(2^{2^{k-1}}) + b2^k & \text{if } k > 0. \end{cases}$$

Let $T_1(k) = T(2^{2^k})$. So, now we have the recurrence as

$$T_1(k) = \begin{cases} d & \text{if } k = 0; \\ 2T_1(k-1) + b2^k & \text{if } k > 0. \end{cases}$$

This recurrence is of the form $T(k) = G(k)T(k-1) + H(k)$ for $k > 1$ with the initial condition $T(0) = d$. As was discussed in the class, this solves to

$$T(k) = (\Pi_{i=1}^{k} G(i)) \left(T(0) + \sum_{i=1}^{k} \frac{H(i)}{\Pi_{j=1}^{i} G(j)} \right).$$

Here, $G(k) = 2$ and $H(k) = b\,2^k$. So $T(k) = 2^k \left(b + \frac{b2^2}{2^2} + \ldots + \frac{b2^k}{2^k} \right) = 2^k(d + bk)$. Now, replace back $2^k = \log n$, to get $T(n) = d \log n + b \log n \log \log n$.